Relativity, Electromagnetism, and Least Action

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1 Introduction

In this lecture we will bring together three ideas which are usually, and unfortunately, taught separately: special relativity, electromagnetism, and the least action principle. First we will review the least action principle, and recall how the equations of motion in pre-relativistic Newtonian mechanics can be derived from a scalar Lagrangian. The connection between the symmetries of space and the dynamically conserved quantities will be evident in this case, and will lead to the statement of Noether's theorem. Next, we look at the conditions which an action for special relativity must satisfy, and find that the choices are greatly limited by the relativity principle; using an appropriate action, we will derive the dynamics of special relavitity, paying special attention to the application of Noether's theorem in finding conserved quantities.

We will then take up the question of how an electromagnetic field affects a charged particle. We will define the four-potential, and see how it enters into the Lagrangian in a natural way, producing the equations of motion we expect. This will lead naturally to the Faraday tensor, the object which reduces to the familiar electric and magnetic fields in the low-velocity limit. We will use the Faraday tensor to write an action for the electromagnetic field itself, and see that the Euler-Lagrange equations express all of Maxwell's laws in a single tensor equation.

2 Noether's Theorem

We start with a brief review. The key points to understand in the Lagrangian formulation of classical mechanics are the following:

1. If we wish to minimize an integral of the form $\int_a^b L(x(t), \dot{x}(t))\, dt$ by choosing a function $x(t)$, we must solve the following differential equation, known as the *Euler-Lagrange equation*:

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0.$$

2. If we wish to minimize a similar integral which depends on a collection of functions $x^i(t)$, the corresponding set of Euler-Lagrange equations is

$$\frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} = 0.$$

The right hand side is a covariant vector. Explicitly, if we make a coordinate transformation $\tilde{x}^j(x^i)$ (where this notation indicates a new set of coordinate functions $\tilde{x}^j$, each depending on the original coordinates), then

$$\frac{\partial L}{\partial \tilde{x}^j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\tilde{x}}^j} = \frac{\partial x^i}{\partial \tilde{x}^j}\left(\frac{\partial L}{\partial \tilde{x}^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\tilde{x}}^i}\right).$$

3. The Lagrangian $L = T - V$, where T and V are the kinetic and potential energies of a system, gives Euler-Lagrange equations which reproduce the equations of motion of classical mechanics.

A brief look at the Euler-Lagrange equations reveals an important point: if the Lagrangian is independent of some coordinate q, such that $\frac{\partial L}{\partial q} = 0$, then $\frac{\partial L}{\partial \dot{q}} = \text{const}$. The quantity $p_q \equiv \frac{\partial L}{\partial \dot{q}}$ is known as the *canonical momentum*, for reasons that will become clear shortly. Let's see how this plays out in a few cases:

1. **Cartesian coordinates, Lagrangian independent of spatial coordinate** $q = x, y$, **or** z: Since $L = T - V$, and the kinetic energy T is a sum of terms $\frac{1}{2}m\dot{x}^2$, the conserved quantity is $p_q = \frac{\partial L}{\partial \dot{q}} = m\dot{q}$. This is the familar linear momentum, which we know to be conserved in appropriate circumstances. Moreover, the Lagrangian formulation makes those circumstances clear: we need L to be independent of q, and therefore V to be independent of q, but this is just the condition that there are no external forces in the q direction.

2. **Spherical coordinates, Lagrangian independent of azimuthal coordinate** ϕ: The kinetic energy in spherical coordinates is still the sum of orthogonal components, but now expressed in the angle coordinates:
$$T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\right).$$
Therefore, if the Lagrangian is independent of ϕ – which you can verify corresponds to a case of no external torque about the z axis – then $p_\phi = mr^2\sin^2\theta\dot{\phi}$ is conserved. This is the angular momentum about the z axis, which is exactly what we would expect to be conserved in such a case.

These examples show that symmetries of a problem generate conserved quantities. Additionally, it shows the conserved quantities which arise in a more elementary treatment of classical mechanics, the linear and angular momentum, correspond precisely to translational and rotational symmetry of space[1]. The only difficulty in this account was the need to change coordinates to see both conservation laws, even though translational and rotational symmetry of space were manifest from the outset.

To alleviate this difficulty, we need to be able to work with symmetries which are subtler than a lack of dependence on some explicit coordinate. For example, the independence of ϕ in the second example above could have been realized in Cartesian coordinates as a symmetry under the rotation

$$\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

Our question now is what the conserved quantity corresponding to this symmetry would be. This will lead us directly to Noether's theorem.

The rotational symmetry above is an example of a continuous, differentiable symmetry. This is in contrast to discrete symmetries, such as $\boldsymbol{r} \to -\boldsymbol{r}$, or the discrete translation symmetry of a crystal lattice. Such a

[1]There is, of course, a seventh conserved quantity: the energy. Conservation of energy is associated with time translation symmetry. Since time is given a special place in classical mechanics, its associated symmetry is harder to tease out, and would take us too far afield. The reader is invited to consider the quantity $H = \sum p_q\dot{q} - L$, and show that $\frac{dH}{dt} = 0$ when $\frac{\partial L}{\partial t} = 0$. Moreover, it can be easily shown that $H = T + V$, the total energy. The quantity H is called the *Hamiltonian*, and leads to an alternative formulation of classical mechanics which is more useful than the Lagrangian formulation in some settings.

continuous symmetry can be encoded by its derivative with respect to a parameter. In the rotation example, the parameter is θ, and the derivative evaluated at $\theta = 0$ is

$$\frac{d}{d\theta}\begin{pmatrix} x \\ y \end{pmatrix}\Bigg|_{\theta=0} = \begin{pmatrix} -y \\ x \end{pmatrix}.$$

We label this derivative as the perturbation vector δq^i. We have been neglecting the z component here; the full perturbation vector is $\delta q^i = (-q^2, q^1, 0) = (-y, x, 0)$.

We now take advantage of the coordinate invariance of the Euler-Lagrange equations. If we transformed to coordinates where $\delta\tilde{q}^i = (1, 0, 0)$, then this would be the symmetry of a single coordinate change, which we have dealt with already. We know that in this case, $\tilde{p}_{\tilde{q}} = \frac{\partial L}{\partial \dot{\tilde{q}}}$ is the conserved quantity. We could write this alternatively as

$$\tilde{p}_{\tilde{q}} = \frac{\partial L}{\partial \dot{\tilde{q}}^i}\delta\tilde{q}^i,$$

where the sum is trivial because of the form of $\delta\tilde{q}^i$. But in this form, we see that $\tilde{p}_{\tilde{q}}$ is an invariant scalar, since it is formed as the contraction of the covariant vector $\frac{\partial L}{\partial \dot{\tilde{q}}^i}$ with the contravariant vector $\delta\tilde{q}^i$. Therefore, if we write the same expression in the original coordinates, we will have exactly the same quantity. The conserved quantity is thus

$$\tilde{p}_{\tilde{q}} = \frac{\partial L}{\partial \dot{q}^i}\delta q^i.$$

We can check our result for the case of the rotation about the z axis. We have

$$\frac{\partial L}{\partial \dot{q}^i}\delta q^i = (m\dot{x})(-y) + (m\dot{y})(x) = m(x\dot{y} - y\dot{x}).$$

Remarkably, we again obtain the z component of angular momentum, this time in Cartesian coordinates.

To emphasize the importance of what we have found, we explicitly state the Noether theorem.

Theorem 2.1 (Noether's Theorem)**.** *Let $L(q^i, \dot{q}^i)$ be a Lagrangian for a system governed by the least action principle. Let δq^i be a vector field for which the solution $q^i(\theta)$ to the differential equation*

$$\frac{dq^i}{d\theta} = \delta q^i$$

satisfies $L(q^i(\theta), \dot{q}^i(\theta)) = L(q^i(0), \dot{q}^i(0))$. Then the Noether charge

$$Q = \frac{\partial L}{\partial \dot{q}^i}\delta q^i$$

is a dynamically conserved quantity, satisfying $\frac{dQ}{dt} = 0$.

The reader is encouraged to derive the conservation of momentum in an arbitrary direction $\hat{\boldsymbol{n}}$, and conservation of angular momentum about the same arbitrary axis, using the Noether theorem[2].

[2] It is also possible to derive the conservation of energy, after upgrading the theorem slightly. Show that if $\frac{\partial L}{\partial q} = \frac{dA}{dt}$, then $p_q - A$ is a conserved quantity. Then follow the logic above to deduce what the Noether charge should be for a vector field δq^i such that $\frac{d}{d\theta}(L(q^i(\theta), \dot{q}^i(\theta))) = \frac{dA}{dt}$. Finally, find a vector field δq^i which corresponds to time translation, and use your extension of the Noether theorem to deduce that the Hamiltonian is the Noether charge for time translation.

3 Relativistic Action and Dynamics

Having seen the great utility of the Lagrangian formulation of classical mechanics for deriving conserved quantities, we will use it to explore less familiar terrain, the dynamics of special relativity. Our first task is to find a suitable action to describe this theory. We will start with a free particle, not under the influence of any forces. The difficulty, in this case, is that time does not have any special place as a parameter in relativity theory, so it would be inappropriate to use an action of the form

$$S = \int_{t_0}^{t_1} L(x^i, \dot{x}^i)\, dt.$$

Rather, we must use the four-vector coordinates x^μ, and integrate over some parameter. A natural choice for the parameter is τ, the proper time. This is the proper time, or the time measured by the particle; in observer coordinates, it satisfies

$$c^2\, d\tau^2 = c^2\, dt^2 - dx^2 - dy^2 - dz^2 = -ds^2.$$

Additionally, the Lagrangian must be a relativistic scalar, invariant under Lorentz transformations, so that the theory itself can have the same symmetry.

There is a natural candidate meeting both criteria. We built special relativity around the assumption that

$$dx^\mu\, dx^\mu = \eta_{\mu\nu}\, dx^\mu\, dx^\nu = -c^2\, dt^2 + dx^2 + dy^2 + dz^2$$

is invariant. Therefore, the quantity $\frac{dx^\mu}{d\tau}\frac{dx_\mu}{d\tau}$ is invariant. Moreover, after multiplying by a factor of $\frac{m}{2}$, this bears a pleasing resemblence to the kinetic energy, which is the classical Lagrangian for a free particle. We will thus try out the action

$$S = \frac{m}{2} \int \frac{dx^\mu}{d\tau}\frac{dx_\mu}{d\tau}\, d\tau,$$

with Lagrangian $\frac{1}{2} m \frac{dx^\mu}{d\tau}\frac{dx_\mu}{d\tau}$, as a candidate for relativity theory. It turns out this is the correct action to describe the theory.

The Euler-Lagrange equations are

$$\frac{\partial L}{\partial x^\mu} = \frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{x}^\mu}\right),$$

where now $\dot{x}^\mu = \frac{dx^\mu}{d\tau}$. Since the Lagrangian does not depend on any of the coordinates the right hand side vanishes, and we have

$$\frac{d}{d\tau}\left(m\frac{dx_\mu}{d\tau}\right) = 0.$$

Therefore, the proper velocity $\frac{dx_\mu}{d\tau}$ is a constant, as we should expect for a free particle.

We can now form the conserved quantities corresponding to the symmetries of spacetime. In Euclidean space, we had three translations and $\binom{3}{2} = 3$ rotations, for 6 total symmetries (the binomial coefficient arises from choosing two coordinates to mix in the rotation). In Minkowski space, we have four translations and $\binom{4}{2} = 6$ rotations, for 10 total symmetries. The translational symmetries are simple: the conserved quantities are simply the canonical momenta

$$p_\mu = \frac{\partial L}{\partial \dot{q}^\mu} = m\frac{dx_\mu}{d\tau}.$$

We can raise the index with the Minkowski metric, obtaining the contravariant momentum $p^\mu = \eta^{\mu\nu} p_\nu = m\frac{dx^\mu}{d\tau}$. The components are:

$$p^0 = \frac{dx^0}{d\tau} = mc\frac{dt}{d\tau} = \frac{m}{\sqrt{1 - v^2/c^2}}, \qquad p^i = m\frac{dx^i}{d\tau} = \frac{mv^i}{\sqrt{1 - v^2/c^2}}.$$

The spatial components are clearly the relativistic analogues of the classical momenta mv^i. The temporal component, however, is less clear. By using the binomial expansion, we can tease out its meaning:

$$p^0 = mc\left(1 - v^2/c^2\right)^{-1/2} \approx \frac{1}{c}\left(mc^2 + \frac{1}{2}mv^2\right).$$

We therefore label $p^0 \equiv E/c$, where $E \approx mc^2 + \frac{1}{2}mv^2$. Since mc^2 is a constant added to the familiar kinetic energy $\frac{1}{2}mv^2$, it has no effect at low velocities. It is known as the rest mass energy. In summary, the conserved momentum associated with translations is $p^\mu = (E/c, p^i)$, where E is energy and the three-vector p^i is the spatial momentum.

We also have six conserved quantities associated with rotations of spacetime, also known as Lorentz transformations. Similar to the previous case, we will derive all of these in a unified manner, and then investigate how they correspond to classical notions. To use the Noether theorem, we need to find vector fields associated with Lorentz transformations. The Lorentz transformations $\tilde{x}^\mu = \Lambda^\mu_\nu x^\nu$ are defined to preserve the Minkowski metric $\eta_{\mu\nu}$. That is,

$$\Lambda^\mu_\sigma \Lambda^\nu_\tau \eta_{\mu\nu} = \eta_{\sigma\tau}.$$

Now, let $\Lambda^\mu_\nu(\theta) = \delta^\mu_\nu + \theta\epsilon^\mu_\nu + \mathcal{O}(\theta^2)$, where δ^μ_ν is the Kronecker symbol. Expanding to first order in θ, we have

$$\eta_{\sigma\tau} + \theta\left(\epsilon^\mu_\sigma \eta_{\mu\tau} + \epsilon^\nu_\tau \eta_{\sigma\nu}\right) = \eta_{\sigma\tau}.$$

We can cancel the $\eta_{\sigma\tau}$ terms, and then relabel dummy indices and use the symmetry of η to find

$$\epsilon^\mu_\sigma \eta_{\mu\tau} + \epsilon^\mu_\tau \eta_{\mu\sigma} = 0.$$

Finally, since η is responsible for raising and lowering indices, this says $\epsilon_{\sigma\tau} + \epsilon_{\tau\sigma} = 0$, so $\epsilon_{\sigma\tau}$ is antisymmetric.

Now, since $x^\mu \to \Lambda^\mu_\nu(\theta)x^\nu$ is a symmetry, we can use the vector field

$$\delta x^\mu = \frac{d}{d\theta}\left(\Lambda^\mu_\nu(\theta)x^\nu\right)\bigg|_{\theta=0} = \epsilon^\mu_\nu x^\nu.$$

The Noether charge is

$$Q = \frac{\partial L}{\partial \dot{x}^\mu}\delta x^\mu = m\eta_{\mu\nu}\delta x^\mu \frac{dx^\nu}{d\tau}.$$

Since $\eta_{\mu\nu}\delta x^\mu = \eta_{\mu\nu}\epsilon^\mu_\sigma x^\sigma = \epsilon_{\nu\sigma}x^\sigma$, we in fact have conservation of

$$Q = m\epsilon_{\nu\sigma}\frac{dx^\nu}{d\tau}x^\sigma,$$

where $\epsilon_{\nu\sigma}$ is an antisymmetric tensor. We can take tensors of the form

$$\epsilon_{\nu\sigma} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

as a basis for antisymmetric tensors. Then we find that the quantities

$$L^{\mu\nu} = m\left(x^\mu \frac{dx^\nu}{d\tau} - x^\nu \frac{dx^\mu}{d\tau}\right)$$

are conserved. The three spatial components (L^{23}, L^{13}, L^{12}) form the relativistic analogue of classical angular momentum. The other three components, L^{0i}, are

$$L^{0i} = m\left(ct\frac{dx^i}{d\tau} - x^i\frac{d(ct)}{d\tau}\right) = mc\frac{dt}{d\tau}\left(v^i t - x^i\right).$$

Conservation of these quantities simply states that a particle maintains a constant position in its own rest frame.

4 Relativistic Particle in a Field

So far, we have found that the Lagrangian $L = \frac{m}{2}\frac{dx^\mu}{d\tau}\frac{dx_\mu}{d\tau}$ gives the relativistic dynamics of a free particle. This is analogous to writing $L = T$, the kinetic energy, and finding the dynamics of a free classical particle. While this is correct, all the interesting physics comes when we add a potential and form the full Lagrangian $L = T - V$. Likewise, we need to add a term to our relativistic Lagrangian to see how particles behave when acted on by forces.

We will focus here on only one kind of force: a force generated by an ambient electromagnetic field. This is because electromagnetism is a manifestly relativistic theory, as we will see, and so it fits naturally as an additional term in the relativistic Lagrangian. However, the usual treatment of electromagnetism treats the three-vectors $\boldsymbol{E}$ and $\boldsymbol{B}$ as the fundamental objects. Neither of these are Lorentz-covariant tensors, and so they are inappropriate here. We will proceed to find tensorial objects which can be used to describe electromagnetism, and then we will be able to form the Lagrangian for a particle in a field.

We will make only two physical assumptions in what follows. The first is, essentially, Coulomb's law: a static charge q generates a field[3] $\boldsymbol{E} = \frac{q}{r^2}\hat{\boldsymbol{r}}$, and the force on a test particle with charge q' is $\boldsymbol{F} = q'\boldsymbol{E}$. We will find this law more useful in its differential form; recall that $\boldsymbol{E} = -\boldsymbol{\nabla}\phi$, where ϕ is the electric potential, and

$$\boldsymbol{\nabla}^2\phi = -\boldsymbol{\nabla}\cdot\boldsymbol{E} = -4\pi\rho,$$

where ρ denotes charge density. The second assumption is Lorentz invariance (that is, the same priniciple of relativity which we used to derive the kinematics of relativistic particles).

Now, as an illustration of the importance of relativity in electromagnetism, consider a wire carrying a current. An elementary problem in magnetostatics is to find the magnetic field around this wire. However, we have made no assumptions about magnetism; we have not even claimed that any such thing as a magnetic field exist. Nonetheless, we can determine that a test particle will feel the force that we would expect from the magnetic field circulating around the wire.

First, we need to find some four-vector associated with electric charge, so that we can make Lorentz transformations and understand how the electric properties of materials change. We already know that four-momentum, $p^\mu = \gamma(mc, \boldsymbol{v})$ is a four-vector. If a particle with this four-momentum has some charge-to-mass ratio $\frac{q}{m}$ in its rest frame, then multiplying by this constant gives another four-vector, called the *four-current*:

$$j^\mu = (\rho c, \boldsymbol{j}),$$

where $\boldsymbol{j} = \rho\boldsymbol{v}$ is the spatial current density, or current per unit area.

We will now exploit the transformation properties of a Lorentz vector. In the lab frame, there is no net charge density (since wires are neutral), and the current density is $\boldsymbol{j} = \frac{I}{\pi R^2}\hat{\boldsymbol{n}}$ (where I is the current in the wire and R is its radius). We then make a Lorentz transformation to the frame of a test particle moving with velocity $v\hat{\boldsymbol{n}}$ (parallel to the wire). The new charge density and current density are

$$\rho' c = \frac{\rho c - \frac{v}{c}(\boldsymbol{j}\cdot\hat{\boldsymbol{n}})}{\sqrt{1-v^2/c^2}} = -\frac{1}{\sqrt{1-v^2/c^2}}\frac{Iv}{\pi R^2 c},$$
$$\boldsymbol{j}'\cdot\hat{\boldsymbol{n}} = \frac{\boldsymbol{j}\cdot\hat{\boldsymbol{n}} - \frac{v}{c}(c\rho)}{\sqrt{1-v^2/c^2}} = \frac{1}{\sqrt{1-v^2/c^2}}\frac{I}{\pi R^2}.$$

The important result is the negative charge density inside the wire. We can use our first assumption to find the electric field associated with this negative charge. This is most easily accomplished by taking a cylinder

[3] We are using Gaussian units. Using SI units in electromagnetism is akin to measuring north-south distances in furlongs and east-west distances in Canadian football fields.

around the wire, integrating $\boldsymbol{\nabla} \cdot \boldsymbol{E}' = 4\pi\rho'$, and using the divergence theorem; the result is

$$\boldsymbol{E}'(R) = -\frac{1}{\sqrt{1 - v^2/c^2}} \frac{2Iv}{c} \frac{\hat{\boldsymbol{r}}}{r}.$$

Therefore the force on the test particle of charge q, in its rest frame is $\boldsymbol{F}' = q\boldsymbol{E}'$. Since we are in the rest frame of the particle, coordinate time is equivalent to proper time; however, to write Newton's law in covariant form, we need to use proper time as the parameter. Thus,

$$\frac{d\boldsymbol{p}'}{d\tau} = q\boldsymbol{E}' = -\frac{1}{\sqrt{1 - v^2/c^2}} \frac{2qIv}{c} \frac{\hat{\boldsymbol{r}}}{r}.$$

Crucially, this is perpendicular to the velocity of the particle. It is a simple exercise to show that transverse components of momentum do not change in a Lorentz transformation. Thus, $\frac{d\boldsymbol{p}'}{d\tau} = \frac{d\boldsymbol{p}}{d\tau}$, where $\boldsymbol{p}$ is the four-momentum in the lab frame. To obtain the force measured in the lab frame, we need to replace proper time with coordinate time, which is accomplished with the chain rule:

$$\boldsymbol{F} = \frac{d\boldsymbol{p}}{dt} = -\frac{2qIv}{c} \frac{\hat{\boldsymbol{r}}}{r}.$$

Thus, a positively charged particle traveling in the same direction as the current is attracted to the wire. This is qualitatively what we would expect from the Lorentz force of a magnetic field circulating around the wire. It is left to the reader to verify that the force has the correct magnitude as well.

In summary, we have shown that the magnetic field in one frame is equivalent to the magnetic field in another frame. This is our first clue that electricity and magnetism are not really different things at all; in a Lorentz-invariant world, we cannot have electricity without magnetism, nor magnetism without electricity.

We will now proceed with the construction of the full theory. We have found a useful four-vector $j^\mu = (c\rho, \boldsymbol{j})$, but this vector tells us properties of matter, and not of the electromagnetic field. Usually the electromagnetic field is written as vectors $\boldsymbol{E}$ and $\boldsymbol{B}$; as already noted, this is inappropriate here. However, the electric field can be written in terms of the electric potential ϕ, which we have already noted is related to charge density by $\boldsymbol{\nabla}^2\phi = -4\pi\rho = -\frac{4\pi}{c}j^0$. This is an idea we can work with; what if we attempt the same construction with all components of the four-current? We would obtain a vector A^μ, satisfying $\boldsymbol{\nabla}^2 A^\mu = -\frac{4\pi}{c}j^\mu$. But this equation is not Lorentz-invariant; the offending party is $\boldsymbol{\nabla}^2$. The correct way to form a relativistic Laplacian operator is to take the gradient vectors

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right), \qquad \partial^\mu = \frac{\partial}{\partial x_\mu} = \left(-\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

and contract them. The result is $\partial_\mu\partial^\mu = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$. Since our assumption of the Coulomb/Gauss law only referred to electrostatic situations, we certainly could have missed a time derivative term in the equations, so it may be the case that $A^0 = \phi$, the electric potential, and that $\partial_\nu\partial^\nu A^\mu = -\frac{4\pi}{c}j^\mu$.

To see if this makes sense, we need to determine the significance of the spatial part of $\boldsymbol{A}$ in $A^\mu = (\phi, \boldsymbol{A})$. We will take a situation of constant[4] $j^\mu = (c\rho, \boldsymbol{j})$, so that

$$\partial_\nu\partial^\nu A^\mu = \boldsymbol{\nabla}^2 A^\mu = -\frac{4\pi}{c}j^\mu.$$

The solution to this equation is familiar from electrostatics; we have

$$A^0 = \phi = \int \frac{\rho(\boldsymbol{r'})\, d\boldsymbol{r'}}{|\boldsymbol{r'} - \boldsymbol{r}|}, \qquad A^i = \frac{1}{c}\int \frac{j^i(\boldsymbol{r'})\, d\boldsymbol{r'}}{|\boldsymbol{r'} - \boldsymbol{r}|}.$$

[4] A constant four-current allows an analogy with electrostatics and an easy interpretation of A^μ, but the equation is not much more difficult to solve in the general case. You can verify that the solution to $\partial_\nu\partial^\nu\phi = -\frac{4\pi}{c}\rho$ is $\phi(\boldsymbol{r}, t) = \int \frac{\rho(\boldsymbol{r'}, t_r)}{|\boldsymbol{r'} - \boldsymbol{r}|}\, d\boldsymbol{r'}$, where $t_r = t - \frac{|\boldsymbol{r} - \boldsymbol{r'}|}{c}$ is the so-called retarded time, the time at which light would have had to leave $\boldsymbol{r'}$ to reach $\boldsymbol{r}$ at time t.

The significance of $\boldsymbol{A}$ is still not clear. We can reveal its meaning by taking the curl. Note that we are differentiating with respect to the coordinates $\boldsymbol{r}$, which are not related to the integration variables $\boldsymbol{r'}$. We obtain

$$\boldsymbol{\nabla}\times\boldsymbol{A}=\frac{1}{c}\int\boldsymbol{\nabla}\times\left(\frac{\boldsymbol{j}(\boldsymbol{r'})}{|\boldsymbol{r'}-\boldsymbol{r}|}\right)\,d\boldsymbol{r'}.$$

You can prove to yourself the identity $\boldsymbol{\nabla}(f\boldsymbol{v})=(\boldsymbol{\nabla}f)\times\boldsymbol{v}+f(\boldsymbol{\nabla}\times\boldsymbol{v})$. In this case $\boldsymbol{\nabla}\times\boldsymbol{j}(\boldsymbol{r'})=0$ (because $\boldsymbol{j}(\boldsymbol{r'})$ does not depend on $\boldsymbol{r}$), and so we have

$$\boldsymbol{\nabla}\times\boldsymbol{A}=\frac{1}{c}\int\boldsymbol{\nabla}\left(\frac{1}{|\boldsymbol{r'}-\boldsymbol{r}|}\right)\times\boldsymbol{j}(\boldsymbol{r'})\,d\boldsymbol{r'}=\frac{1}{c}\int\frac{(\boldsymbol{r'}-\boldsymbol{r})\times\boldsymbol{j}(\boldsymbol{r'})}{|\boldsymbol{r'}-\boldsymbol{r}|^3}\,d\boldsymbol{r'}.$$

At last we have a recognizable result: the right hand side is the Biot-Savart law for the magnetic field $\boldsymbol{B}$. Thus, $\boldsymbol{\nabla}\times\boldsymbol{A}=\boldsymbol{B}$.

We are now confident that the four-potential A^μ contains information about the electric and magnetic fields. Moreover, at least in its temporal component $A^0=\phi$, we have the intuition that it is related to potential energy. It is thus a very promising candidate for incorporation into the relativistic Lagrangian. However, we can only add scalars to the Lagrangian, so we need to contract it with some vector. This vector ought to be related to the particle dynamics, so that the resulting Euler-Lagrange equations relate the electromagnetic field to the motion of the particle. A natural choice is the four-velocity. Therefore, our scalar of interest is $\frac{\partial x_\mu}{\partial\tau}A^\mu$. There may also be an overall constant, so our candidate Lagrangian is

$$L=\frac{m}{2}\frac{\partial x^\mu}{\partial\tau}\frac{\partial x_\mu}{\partial\tau}+\eta\frac{\partial x_\mu}{\partial\tau}A^\mu,$$

where η is yet to be determined. We can set η by looking at the classical limit with $v\ll c$. In this limit $\frac{\partial x^0}{\partial\tau}\approx c$ and $\frac{\partial x^i}{\partial\tau}\approx v^i\ll c$, so

$$L\approx-\frac{mc^2}{2}+\frac{mv^2}{2}-\eta c\phi.$$

The first term is a constant and does not contribute. The second term is the familiar kinetic energy, and the third term ought to be a potential energy; for this to be the case, we must have $\eta=\frac{q}{c}$. This determines our Lagrangian:

$$L=\frac{m}{2}\frac{\partial x^\mu}{\partial\tau}\frac{\partial x_\mu}{\partial\tau}+\frac{q}{c}\frac{\partial x_\mu}{\partial\tau}A^\mu.$$

Now we can determine the dynamics of particles in an electromagnetic field, using the Euler-Lagrange equations. We no longer have $\frac{\partial L}{\partial x^\mu}=0$, since the four-potential A^μ is a function of the coordinates x^μ. Thus, the equations are

$$\frac{\partial L}{\partial x^\mu}-\frac{d}{d\tau}\frac{\partial L}{\partial\dot{x}^\mu}=\frac{q}{c}\frac{\partial x_\nu}{\partial\tau}\frac{\partial A^\nu}{\partial x^\mu}-\frac{\partial}{\partial\tau}\left(m\frac{\partial x^\mu}{\partial\tau}+\frac{q}{c}A^\mu\right)=0.$$

We can dissect this equation a great deal. First, we distribute the proper time derivative and rearrange, so we have something resembling a force law:

$$m\frac{d^2x_\mu}{d\tau^2}=\frac{q}{c}\left(\frac{\partial x_\nu}{\partial\tau}\frac{\partial A^\nu}{\partial x^\mu}-\frac{dA_\mu}{d\tau}\right).$$

Raising indices and separating the temporal and spatial parts of A^μ, we have

$$m\frac{d^2x^\mu}{d\tau^2}=q\frac{dt}{d\tau}\left(-\frac{\partial\phi}{\partial x_\mu}-\frac{1}{c}\frac{dA^\mu}{dt}+\frac{\boldsymbol{v}}{c}\cdot\frac{\partial\boldsymbol{A}}{\partial x^\mu}\right).$$

We are interested in the spatial component of the left hand side, which is related to the force. The spatial part is

$$m\frac{d^2\boldsymbol{x}}{d\tau^2}=q\frac{dt}{d\tau}\left(-\boldsymbol{\nabla}\phi-\frac{1}{c}\frac{d\boldsymbol{A}}{dt}+\frac{1}{c}\boldsymbol{\nabla}(\boldsymbol{v}\cdot\boldsymbol{A})\right).$$

You can verify to yourself the following useful vector identity:

$$\boldsymbol{v} \times (\boldsymbol{\nabla} \times \boldsymbol{A}) = \boldsymbol{\nabla}(\boldsymbol{v} \cdot \boldsymbol{A}) - (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{A}.$$

Additionally, recall that

$$\frac{d\boldsymbol{A}}{dt} = \frac{\partial \boldsymbol{A}}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{A};$$

this is just the multivariable chain rule. Therefore,

$$-\frac{1}{c}\frac{d\boldsymbol{A}}{dt} + \frac{1}{c}\boldsymbol{\nabla}(\boldsymbol{v} \cdot \boldsymbol{A}) = -\frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} + \frac{1}{c}\left(\boldsymbol{\nabla}(\boldsymbol{v} \cdot \boldsymbol{A}) - (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{A}\right) = \frac{\boldsymbol{v}}{c} \times (\boldsymbol{\nabla} \times \boldsymbol{A}).$$

Thus, the spatial part of the Euler-Lagrange equations become

$$m\frac{d^2\boldsymbol{x}}{d\tau^2} = q\frac{dt}{d\tau}\left(-\boldsymbol{\nabla}\phi - \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} + \frac{\boldsymbol{v}}{c} \times (\boldsymbol{\nabla} \times \boldsymbol{A})\right).$$

In the same way a with the example of the wire, we can change the parameter from τ to t on the left at the cost of a factor of $\frac{d\tau}{dt}$ on the right; this cancels the existing factor of $\frac{dt}{d\tau}$, and so

$$\boldsymbol{F} = m\frac{d^2x}{dt^2} = q\left(-\boldsymbol{\nabla}\phi - \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} + \frac{\boldsymbol{v}}{c} \times (\boldsymbol{\nabla} \times \boldsymbol{A})\right).$$

If we define

$$\boldsymbol{E} = -\boldsymbol{\nabla}\phi - \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t}, \qquad \boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$$

then this is the familiar Lorentz force law.

It is now clear that $\boldsymbol{E}$ and $\boldsymbol{B}$ are unfortunate objects, not being Lorentz covariant. However, we can define an object which contains them both and does have the required covariance properties: the *Faraday tensor*

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu.$$

Using the definitions of $\boldsymbol{E}$ and $\boldsymbol{B}$ above, and recalling that $\partial^0 = -\frac{1}{c}\frac{\partial}{\partial t}$, we find that the components of this tensor are

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix}.$$

Remember that Lorentz transformations at low velocities (in comparison to c) barely mix the temporal and spatial components of tensors. Thus, the components we have labeled as E_i barely mix with the components we have labeled as B_i. This is the reason we refer to $\boldsymbol{E}$ and $\boldsymbol{B}$ as separate vectors: at our low velocities, they seem to be. It is only our sloggish speeds which prevented for some time the realization that the electromagnetic field is a single tensor object, $F^{\mu\nu}$. In the next section, we will use this tensor to find a Lagrangian for the electromagnetic field itself, and use it to derive the Maxwell equations.

5 Lagrangian Field Theory

So far, we have used the least action principle to deduce the dynamics of particles. Particles are described classically by a vector at each point in time, or relativistically by a four-vector at each value of their own proper time. In each case, there is a single parameter. A field is different; it has a value at every point in time and in space, so it has four parameters. An appropriate action must integrate over all these parameters. We define a functional $\mathcal{L}$, the *Lagrangian density*, to be the integrand in this action. Then the Lagrangian itself is

$$L = \int \mathcal{L}\, d^3\boldsymbol{x},$$

and the action is

$$S = \int L\, dt = \iint \mathcal{L}\, d^3\boldsymbol{x}\, dt = \int \mathcal{L}\, d^4x,$$

where $x^\mu = (t, \boldsymbol{x})$ is a four-vector for position in spacetime.

We also need to specify what the Lagrangian density $\mathcal{L}$ depends on. For particles, the Lagrangian depended on the function which gives the position of a particle at each time, and its derivative. Likewise, for fields, the Lagrangian density depends on the function giving the value (whether it is a scalar, vector, or higher rank tensor) of the field at each point in spacetime; but this is just the field itself. It also depends on all the first derivatives of this field. So, for a scalar field ϕ, we would have

$$S = \int \mathcal{L}(\phi, \partial^\mu \phi)\, d^4x.$$

Our goal here will be to develop the theory of the electromagnetic field. It will turn out that a vector field will aptly describe this theory, so we will focus on vector fields from now on. For a vector field χ^μ, the Lagrangian density has the form $\mathcal{L}(\chi^\mu, \partial^\nu \chi^\mu)$.

When we derived the Euler-Lagrange equations, we discretized a time interval and varied a function y at each point, and demanded that the action be stationary with respect to all these variations. The variation of S came from the change in the function value itself, which led to a $\frac{\partial L}{\partial y}$ term in the equations, and the change in the derivative, which led to a $-\frac{d}{dt}\frac{\partial L}{\partial \dot{y}}$ term. In the case of a field, the logic is similar, except that changing the field value at a point changes each component of its gradient. We should therefore replace the time derivative term with a sum over all derivatives. The result for the case of a vector field is

$$\frac{\partial \mathcal{L}}{\partial \chi^\mu} - \partial^\nu \frac{\partial \mathcal{L}}{\partial(\partial^\nu \chi^\mu)} = 0.$$

Note that μ is a free index in this expression, while ν is summed over. This equation thus says that a vector, which we again call the Euler-Lagrange vector, vanishes. Crucially, this vector is generally covariant, and so the Euler-Lagrange equations can be written in any coordinates.

We will now construct a Lagrangian density for the electromagnetic field. We start with three conditions: it must be a Lorentz scalar; it must be related to the familiar $\boldsymbol{E}$ and $\boldsymbol{B}$ fields so that we can eventually extract Maxwell's equations; and it must have units of energy density. The first and second considerations suggest that we should be looking at an expression involving $F^{\mu\nu}$ with the indices contracted against something. The third consideration suggests that we should have two powers of the Faraday tensor, since the Faraday tensor is linear in the electric and magnetic fields and we know E^2 and B^2 have units of energy density (in Gaussian units). The most natural choice is $\mathcal{L} = F_{\mu\nu}F^{\mu\nu}$. This satisfies all three desired properties: it is manifestly a scalar, it is related to the familiar electric and magnetic fields, and it has the appropriate units. We will also need a term related to matter, so that we can obtain equations relating the fields to their sources; first, though, we will investigate this Lagrangian on its own, and obtain Maxwell's equations in vacuum.

Once we form the Euler-Lagrange equations, we will see it is most natural to instead choose $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$. The overall constant does not affect this Lagrangian, but it will be convenient later on, so for clarity we will introduce it from the beginning. Written in terms of the vector field A^μ, the Lagrangian density is

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}\eta_{\lambda\mu}\eta_{\rho\nu}(\partial^\lambda A^\rho - \partial^\rho A^\lambda)(\partial^\mu A^\nu - \partial^\nu A^\mu)\\ &= -\frac{1}{4}\eta_{\lambda\mu}\eta_{\rho\nu}(\partial^\lambda A^\rho \partial^\mu A^\nu - \partial^\rho A^\lambda \partial^\mu A^\nu - \partial^\lambda A^\rho \partial^\nu A^\mu + \partial^\rho A^\lambda \partial^\nu A^\mu).\end{aligned}$$

The Euler-Lagrange equations then tell us

$$\frac{\partial \mathcal{L}}{\partial A^\mu} - \partial^\nu\left(\frac{\partial \mathcal{L}}{\partial(\partial^\nu A^\mu)}\right) = -\partial^\nu\left(\frac{\partial \mathcal{L}}{\partial(\partial^\nu A^\mu)}\right) = 0.$$

The term in parentheses can be computed directly from the extended expression above for $\mathcal{L}$. This is an excellent exercise in index manipulation. When we differentiate with respect to $\partial^{\nu'} A^{\mu'}$ (with primes for clarity), the first term gives

$$\frac{\partial}{\partial(\partial^{\nu'} A^{\mu'})}\left(\eta_{\lambda\mu}\eta_{\rho\nu}\partial^\lambda A^\rho \partial^\mu A^\nu\right) = \eta_{\nu'\mu}\eta_{\mu'\nu}\partial^\mu A^\nu + \eta_{\lambda\nu'}\eta_{\rho\mu'}\partial^\lambda A^\rho = 2\partial_{\nu'} A_{\mu'}.$$

Work through this and ensure you understand it. The other terms follow in exactly the same way, and we have

$$\frac{\partial \mathcal{L}}{\partial(\partial^\nu A^\mu)} = -\frac{1}{4}\left(2\partial_\nu A_\mu - 2\partial_\mu A_\nu - 2\partial_\mu A_\nu + 2\partial_\nu A_\mu\right) = F_{\mu\nu}.$$

At this point the motivation for the factor of $-\frac{1}{4}$ should be evident. The equations of motion for the field are

$$\partial^\nu F_{\mu\nu} = 0, \qquad \text{or equivalently,} \qquad \partial_\nu F^{\mu\nu} = 0.$$

We can write out this equation in terms of the familiar fields. The first component is

$$\partial_\nu F^{0\nu} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \boldsymbol{\nabla}\cdot\boldsymbol{E} = 0.$$

This is Gauss's law in a vacuum. The x component is

$$\partial_\nu F^{1\nu} = -\frac{1}{c}\frac{\partial E^x}{\partial t} + \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} = 0.$$

If we collect this together with the other components, we find

$$\boldsymbol{\nabla}\times\boldsymbol{B} = \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t},$$

which is Ampere's law in a vacuum (the right hand side is the so-called displacement current).

This is promising, but we seem to be missing two of Maxwell's laws. In fact, we are not: they are contained in our definitions of the fields. Since $\boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A}$, we automatically have $\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0$. And, since

$$\boldsymbol{E} = -\boldsymbol{\nabla}\phi - \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t},$$

we automatically have

$$\boldsymbol{\nabla}\times\boldsymbol{E} = -\frac{1}{c}\frac{\partial(\boldsymbol{\nabla}\times\boldsymbol{A})}{\partial t} = -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t},$$

which is Faraday's law.

Now, to add in sources, we need to include a matter Lagrangian. But we already have this:

$$L = \frac{m}{2}\frac{\partial x^\mu}{\partial \tau}\frac{\partial x_\mu}{\partial \tau} + \frac{q}{c}\frac{\partial x_\mu}{\partial \tau}A^\mu.$$

We will not need the first term, since this contains information only about the matter itself, and we are interested in the coupling to the field. We need to write the second term as the integral over space of a Lagrangian density; we can do this by replacing q with charge density ρ, so

$$\mathcal{L} = \frac{1}{c}\left(\rho\frac{\partial x_\mu}{\partial \tau}\right)A^\mu.$$

But the quantity in parentheses is simply j_μ. Thus, by adding this to the field Lagrangian, we can form a total Lagrangian density

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{4\pi}{c}j_\mu A^\mu,$$

where we have multiplied the matter Lagrangian by 4π so that we obtain the correct constant in the resulting equations. The second term only contributes to the $\frac{\partial \mathcal{L}}{\partial A^\mu}$ part of the Euler-Lagrange equations, and its contribution is obvious. We can immediately write the equations of motion of the field:

$$\partial_\nu F^{\mu\nu} = \frac{4\pi}{c}j^\mu.$$

Expanding this into its components, we have Gauss's and Ampere's law in their complete forms:

$$\begin{aligned} \boldsymbol{E} &= 4\pi\rho, \\ \boldsymbol{\nabla}\times\boldsymbol{B} &= \frac{4\pi}{c}\boldsymbol{j} + \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t}. \end{aligned}$$