Undergraduate Physics *in a Hurry*

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$$\frac{d\boldsymbol{p}_i}{dt} = \boldsymbol{F}_i = \sum_{j \neq i} \boldsymbol{F}_{j \to i}$$

$$L(x, \dot{x}) = T - V$$

$$H(x, p) = \sum_x p_x \dot{x} - L$$

$$\boldsymbol{F} = q\left(\boldsymbol{E} + \frac{\boldsymbol{v}}{c} \times \boldsymbol{B}\right)$$

$$\phi(r, \theta) = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right) P_\ell(\cos\theta)$$

$$\nabla \cdot \boldsymbol{E} = 4\pi\rho$$

$$\dot{x} = \frac{\partial H}{\partial p}, \dot{p} = -\frac{\partial H}{\partial x}$$

$$\omega = \sqrt{\frac{1}{m}\left.\frac{d^2V}{dx^2}\right|_{x=x_0}}$$

$$\boldsymbol{F} = -\frac{Gm_1m_2}{r^2}\hat{\boldsymbol{r}}$$

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{1}{c}\int d^3\boldsymbol{r}' \frac{\boldsymbol{j} \times (\boldsymbol{r} - \boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|^3}$$

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}$$

$$\nabla \cdot \boldsymbol{B} = 0$$

$$\nabla^2\phi = 4\pi G\rho$$

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$$

$$\frac{d\boldsymbol{L}}{dt} = \boldsymbol{r} \times \boldsymbol{F}$$

$$\boldsymbol{j} = \sigma\boldsymbol{E}$$

$$\boldsymbol{B} = \boldsymbol{H} + 4\pi\boldsymbol{M}$$

$$\boldsymbol{D} = \boldsymbol{E} + 4\pi\boldsymbol{P}$$

$$\boldsymbol{L} = I\boldsymbol{\omega}$$

$$\boldsymbol{\tau} = I\boldsymbol{\alpha}$$

$$T = \frac{1}{2}I\omega^2$$

$$\nabla \times \boldsymbol{B} = \frac{1}{c}\left(4\pi\boldsymbol{j} + \frac{\partial \boldsymbol{E}}{\partial t}\right)$$

$$\boldsymbol{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t}$$

$$\nabla \times \boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t}$$

$$L^i = I^i{}_j\omega^j$$

$$\frac{d\boldsymbol{A}}{dt} = \left(\frac{d\boldsymbol{A}}{dt}\right)_{\text{rot}} + \boldsymbol{\omega} \times \boldsymbol{A}$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla\boldsymbol{u} = -\frac{\nabla p}{\rho} + \boldsymbol{g}$$

$$\boldsymbol{g} = \frac{1}{c^2}\boldsymbol{S}$$

$$\frac{\partial u}{\partial t} = -\nabla \cdot \boldsymbol{S} - \boldsymbol{E} \cdot \boldsymbol{j}$$

$$u = \frac{1}{8\pi}\left(E^2 + B^2\right)$$

$$\boldsymbol{a}_{\text{rot}} = \boldsymbol{a} - 2\boldsymbol{\omega} \times \boldsymbol{v} - \boldsymbol{\omega} \cdot (\boldsymbol{\omega} \times \boldsymbol{r})$$

$$P = \frac{2q^2a^2}{3c^3}$$

$$f(\theta, \phi) = -\frac{m}{2\pi\hbar^2}\int d^3\boldsymbol{r}'\, e^{-i\boldsymbol{q}\cdot\boldsymbol{r}'} V(\boldsymbol{r}')$$

$$U = \frac{3}{2}pV$$

$$U(t) = T\exp\left(-\tfrac{i}{\hbar}\int_0^t V_I(t')\,dt'\right)$$

$$\frac{d\sigma}{d\Omega} = |f(\theta, \phi)|^2$$

$$|\psi\rangle = |\psi_i\rangle + G(E)V\,|\psi\rangle$$

$$\langle N_i\rangle = \frac{1}{e^{\beta(\epsilon_i - \mu)} - \eta}$$

$$Z = \operatorname{tr}\left(e^{-\beta H}\right)$$

$$\rho = \sum P(\psi)\,|\psi\rangle\,\langle\psi|$$

$$|n^{(1)}\rangle = \sum_{k \neq n} \frac{\langle k^{(0)}|V|n^{(0)}\rangle}{E_n^{(0)} - E_k^{(0)}}\,|k^{(0)}\rangle$$

$$E_n^{(2)} = \sum_{k \neq n} \frac{|\langle k^{(0)}|V|n^{(0)}\rangle|^2}{E_n^{(0)} - E_k^{(0)}}$$

$$c_n(t) = c_n(0)\exp\left(\int_0^t \langle\psi_n(t)|\dot{\psi}_n(t)\rangle\,dt\right)$$

$$\left(p + \tfrac{aN^2}{V^2}\right)(V - Nb) = Nk_BT$$

$$2\gamma k_BT = \int d\tau \langle R(0)R(\tau)\rangle$$

$$\mathcal{G} = -k_BT\log\mathcal{Z}$$

$$H = \mu_B\boldsymbol{B} \cdot \boldsymbol{\sigma}$$

$$j_1 \otimes j_2 = (j_1 + j_2) \oplus (j_1 + j_2 - 1) \oplus \cdots \oplus |j_1 - j_2|$$

$$E_n^{(1)} = \langle n^{(0)}|V|n^{(0)}\rangle$$

$$\mathcal{G} = U - TS - \mu N$$

$$P(\boldsymbol{p}) = \left(\tfrac{1}{2\pi m k_BT}\right)^{3/2}\exp\left(-\tfrac{p^2}{2mk_BT}\right)$$

$$F = -k_BT\log Z$$

$$a^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

$$[J_i, J_j] = i\hbar\epsilon_{ijk}J_k$$

$$J_\pm\,|j, m\rangle = \hbar\sqrt{j(j+1) - m(m\pm1)}\,|j, m\pm1\rangle$$

$$P(U) = \tfrac{1}{Z}\exp\left(-\tfrac{E}{k_BT}\right)$$

$$S = k_B\log\Omega$$

$$S\,dT - V\,dp + N\,d\mu = 0$$

$$i\hbar\frac{d\hat{A}}{dt} = [\hat{A}, \hat{H}] + i\hbar\frac{\partial\hat{A}}{\partial t}$$

$$i\hbar\frac{d}{dt}\,|\psi\rangle = \hat{H}\,|\psi\rangle$$

$$\hat{H}\,|\psi\rangle = E\,|\psi\rangle$$

$$dU = T\,dS - p\,dV$$

$$dS = \frac{đQ}{T}$$

$$\eta = 1 - \frac{T_C}{T_H}$$

Contents

Chapter 1

Mechanics

There's a lot of stuff in the world, and the stuff moves around a lot. How exactly does the stuff move around? Newton thought he knew. It turns out he didn't know, as Einstein made clear. However, Fig. 1.1 shows that Newton had a much bolder hairsytle than Einstein. Out of respect for his wild fashion choice, and for the subject distribution on the preliminary exams, this chapter will be spent on Newtonian mechanics.

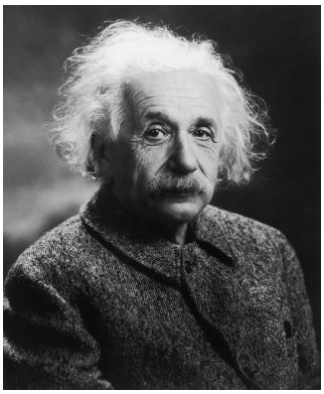


Figure 1.1: On the left, People Magazine's Sexiest Man Alive of 1687, Isaac Newton. On the right, Albert Einstein, wondering why Newton's face is so skinny, and suddenly realizing that Lorentz contraction might be real after all.

In 1.1, we'll review mechanics in its Newtonian, Lagrangian, and Hamiltonian formulations. In 1.2, we focus on stuff that moves back and forth, and describe how it does so. In 1.3, we look at how gravity can make stuff move back and forth, or make stuff bounce, and how different kinds of forces can also make stuff bounce. In 1.4, we begrudgingly accept that not all stuff consists of point particles, and talk about stuff that's big and rigid. In 1.5, we let the real world encroach even further, by talking about stuff that's big and bendy. In 1.6, we talk about how we all live on a yellow submarine, and the submarine can accelerate, leading to apparent forces. Finally, in 1.7, we collect additional interesting problems about stuff that moves.

1.1 Review of Basics

Newton's three laws can be summarized in one statement: momentum is conserved. To spell it out in the usual trifecta:

- If an object is #foreveralone, it has nobody to share its momentum with. It has to keep its own momentum constant, as well as its mass, so it moves with a constant velocity $\boldsymbol{v}$.
- If an object has a friend, it can exchange momentum with that friend. If $\boldsymbol{p}_i$ denotes the momentum of the ith object, then as a convention, we'll denote the rate at which its friends give it momentum by the force $\boldsymbol{F}_i$, where $\boldsymbol{F}_i$ is the sum of the rate of momentum $\boldsymbol{F}_{j\to i}$ given by all friends $j \neq i$. Symbolically,

$$\frac{d\boldsymbol{p}_i}{dt} = \boldsymbol{F}_i = \sum_{j\neq i} \boldsymbol{F}_{j\to i}. \tag{1.1.1}$$

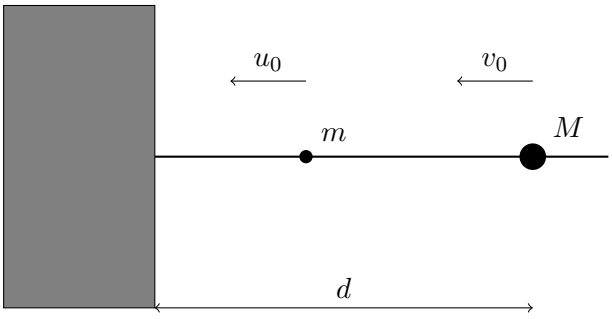
- Friends don't let friends violate momentum conservation. One way to ensure momentum conservation is to require each two-body interaction to be momentum conserving:

$$\boldsymbol{F}_{i\to j} = -\boldsymbol{F}_{j\to i}. \tag{1.1.2}$$

  This will work for now. It won't work in Chapter 2, when the friends are invisible and all around us.

This demonstrates how Newton's laws are *almost* devoid of content. Since they don't tell us the forces themselves, they're rarely enough on their own. But momentum conservation is a powerful statement, and sometimes it's all we need.

Problem 1.1 (M04M1)
Two elastic spherical particles with masses m and M ($m \ll M$) are constrained to move along a straight line with an elastically reflecting wall at its end. At $t = 0$ they are in motion as shown, with $u_0 \gg v_0$.



Find the subsequent motion of M, averaged over the period of motion of m. In this approximation, how far does M travel before turning around?

Let's let $x = 0$ denote the wall position, x and u the small mass position and velocity, and X and v the large mass position and velocity. There are two kinds of collisions. The small mass could bounce off the wall, in which case we obviously have $v \to -v$, $V \to V$. The small mass could also bounce off the large mass, in which case we have to do a bit of work. One could write down conservation of momentum and energy, shift to a center of mass frame, solve, and shift back to determine

$$u \to 2\frac{mu + Mv}{m + M} - u, \qquad v \to 2\frac{mu + Mv}{m + M} - v. \tag{1.1.3}$$

But one would be working too hard. Geometry is the key to this problem. Looking carefully at the forms of momentum and energy,

$$mu + Mv, \qquad \frac{1}{2}mu^2 + \frac{1}{2}Mv^2, \tag{1.1.4}$$

it becomes clear that we should be looking at the vector $\boldsymbol{V} = (\sqrt{m}u, \sqrt{M}v)$. Its magnitude is constant as a consequence of energy conservation, and its dot product with the fixed vector $\boldsymbol{a} = (\sqrt{m}, \sqrt{M})$ is constant as a consequence of momentum conservation. The only way to change $\boldsymbol{V}$ while keeping $\boldsymbol{V} \cdot \boldsymbol{a}$ and $|\boldsymbol{V}|$ constant is to reflect $\boldsymbol{V}$ about $\boldsymbol{a}$, which looks like "bouncing" $\boldsymbol{V}$ off $\boldsymbol{a}$.

Interesting. We started by talking about a 1D bounce, and now we're talking about a 2D bounce. Let's change coordinates and make the bouncing analogy even more precise: let

$$\begin{aligned} x' &= \sqrt{m}x, \qquad & X' &= \sqrt{M}X, \\ u' &= \sqrt{m}u, \qquad & v' &= \sqrt{M}v. \end{aligned} \tag{1.1.5}$$

Now $\boldsymbol{V} = (u', v')$, and it's still bouncing off of $\boldsymbol{a}$. The point of the coordinate change was to (a) make $\boldsymbol{V}$ the velocity in x'-X' space, and to (b) make $\boldsymbol{a}$ parallel to the boundary of the configuration space, as shown in Figure 1.2.

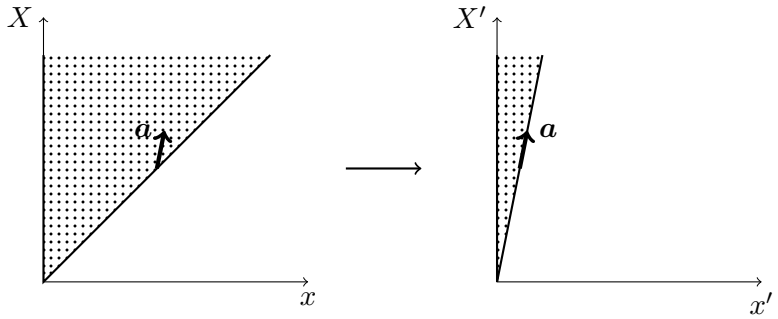


Figure 1.2: The accessible region of configuration space is shaded in both coordinate systems. The coordinate change (1.1.5) brings the oblique edge into alignment with $\boldsymbol{a}$.

With both of these things settled, we now have a problem which looks very different from what we started with: a billiards ball bouncing around in a narrow triangle. There's a well-known approach for billiards problems like this: "unwrap" the triangle into its successive mirror images, and let the billiards ball move in a straight line through the hall of mirrors. This is shown in Figure 1.3.

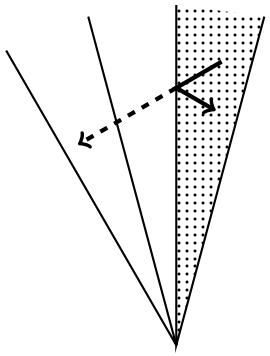


Figure 1.3: By reflecting the triangle about its edges, we can unravel the billiards trajectory into a straight line.

Since $m \ll M$, the triangles are tall and skinny, so we can use the distance from the origin in the unraveled trajectory as a proxy for X'. The problem boils down to determining how close to the origin we get if we start out at $(0, \sqrt{M}d)$ and start moving in the $(-\sqrt{m}u_0, -\sqrt{M}v_0)$ direction. A pinch of trigonometry gives an answer of

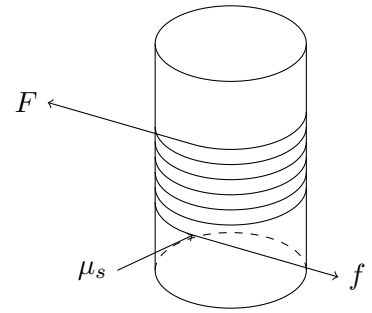
$$X'_{\min} \approx \sqrt{M}d \cos\left(\tan^{-1}\frac{\sqrt{M}v_0}{\sqrt{m}u_0}\right) = \sqrt{M}d\frac{\sqrt{m}u_0}{\sqrt{Mv_0^2 + mu_0^2}}. \tag{1.1.6}$$

It follows that the large mass moves a distance

$$d - \frac{X'_{\min}}{\sqrt{M}} = d\left(1 - \frac{1}{\sqrt{1 + \frac{Mv_0^2}{mu_0^2}}}\right). \tag{1.1.7}$$

That was smooth, if I may say so myself. But life isn't always so smooth. Sometimes the going gets rough, and then energy is dissipated due to friction. Friction is a complicated area of modern research, but in undergraduate mechanics, we typically ignore this and model friction according to Amonton's laws. The force of static friction can have a magnitude up to $\mu_s N$, where μ_s is the coefficient of static friction and N is the magnitude of the normal force. Kinetic friction follows the same rule with μ_k in place of μ_s.

Problem 1.2 (J05M1)
A long rope is wound around a cylinder of radius r so that a length, l, of the rope is in contact with the cylinder. The coefficient of static friction between the rope and the cylinder is μ_s. A force F is exerted on one end of the rope. For a given F, r, l, and μ_s, what force f must be applied to avoid the rope slipping? Explain why a small child can hold a large ocean liner in place using a device like this.



This is a classic. The solution follows immediately from the rule for static friction. If we look at an infinitesimal angular slice $d\theta$, where the tension on the rope is $T(\theta)$, the normal force is $2T(\theta)\sin\frac{d\theta}{2} \approx T(\theta)d\theta$. That means static friction is permitted to assist our small child by up to $\mu_s T(\theta)d\theta$ of force. It follows that

$$dT = \mu_s T\, d\theta, \tag{1.1.8}$$

and integrating this gives $T = fe^{\mu_s\theta} = fe^{\mu_s l/r}$. In order for this tension to balance the large force F, we only need $f = Fe^{-\mu_s l/r}$, so a few turns of rope around the cylinder will suffice to exponentially suppress the force of the ocean liner.

When it comes to friction, Newton's laws are the way to go. But when it comes to just about anything else, there are fancier tools waiting to be used. These tools reflect in a deep way on the structure of physical theories and on the link between classical and quantum physics, but they're also a nice way to solve exam problems quickly.

Lagrangian mechanics starts from a scalar function, the Lagrangian, which depends on positions and velocities. In mechanics, it is given by the difference between kinetic and potential energies:

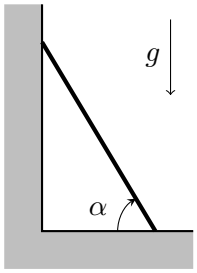
$$L(x, \dot{x}) = T - V. \tag{1.1.9}$$

We can form the equations of motion using the Euler-Lagrange equations,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0. \tag{1.1.10}$$

The main attraction of this method is that the coordinates x need not be the Cartesian position coordinates. The Euler-Lagrange equations enjoy a general coordinate invariance, so we can use any coordinates we damn well please, and Euler-Lagrange will spit out the correct equations of motion. This is especially useful for constrained systems.

Problem 1.3 (J09M3)
A uniform ladder leans against a frictionless vertical wall and rests on a frictionless horizontal floor. It is released from rest, with the ladder and the floor initially making an angle α. At some point, the ladder will separate from the wall. Determine the angle the ladder makes with the floor when this happens.



The only horizontal force on the ladder is the normal force from the wall, N_w, so

$$N_w = m\ddot{x}, \tag{1.1.11}$$

where x is the horizontal coordinate of the center of mass of the ladder, and m is its mass. In order for the ladder to separate from the wall, we need $N_w = 0$, or $\ddot{x} = 0$.

Now we determine $\ddot{x}$. But first, we recognize that x is a rather unfortunate coordinate to be using, because there's also a y, and x and y are constrained to lie on some circular arc and that's messy. So instead, we'll let θ be our coordinate, where θ is the angle between the ladder and the floor. The center of mass of the ladder is at $\left(\frac{\ell}{2}\cos\theta, \frac{\ell}{2}\sin\theta\right)$, where ℓ is the length of the ladder. From this, we can immediately write down a Lagrangian,

$$L = \frac{m\ell^2}{8}\dot{\theta}^2 - \frac{mg\ell}{2}\sin\theta. \tag{1.1.12}$$

And in turn, we can immediately write down the Euler-Lagrange equation,

$$\ddot{\theta} - \frac{2g}{\ell}\cos\theta = 0. \tag{1.1.13}$$

Look familiar? It's the equation of motion for an inverted pendulum with length $\frac{\ell}{2}$ (and a shifted angle coordinate). That's because the center of mass is constrained to lie on a circle with this radius. Anyway, we want to solve $\ddot{x} = 0$, so let's compute:

$$\ddot{x} = \frac{d}{dt}\left(-\frac{\ell}{2}\dot{\theta}\sin\theta\right) = \frac{\ell}{2}\left(-\ddot{\theta}\sin\theta - \dot{\theta}^2\cos\theta\right). \tag{1.1.14}$$

We now have two equations for $\ddot{\theta}$, so we can eliminate and find

$$\frac{2g}{\ell}\sin\theta - \dot{\theta}^2 = 0. \tag{1.1.15}$$

Finally, we use conservation of energy. The energy is

$$\frac{m\ell^2}{8}\dot{\theta}^2 + \frac{mg\ell}{2}\sin\theta, \tag{1.1.16}$$

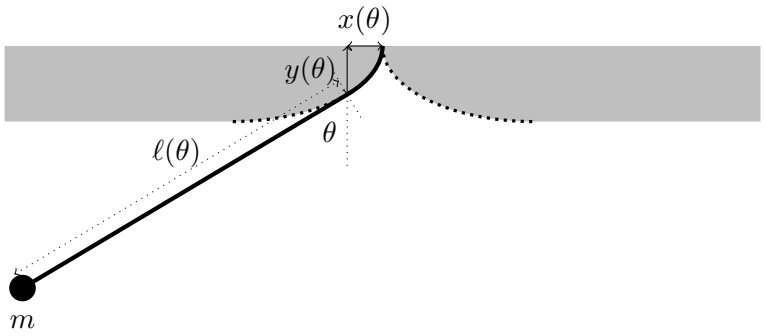
and it will always be equal to its initial value, $\frac{mg\ell}{2}\sin\alpha$. This gives

$$\dot{\theta}^2 = \frac{4g}{\ell}(\sin\alpha - \sin\theta), \tag{1.1.17}$$

so we can solve and find

$$\theta = \sin^{-1}\left(\frac{2}{3}\sin\alpha\right). \tag{1.1.18}$$

Problem 1.4 (M06M2)
To compensate for the fact that the period of a simple pendulum depends on the amplitude of oscillation, the 17$^{\text{th}}$ century Dutch physicist Christian Huygens devised the following setup, depicted in the figure below. It shows a simple pendulum consisting of a mass m and a string of length ℓ_0 whose motion is constrained by a cusp shaped piece of wood. The problem is to determine the shape of the wooden surface so that the period of the pendulum is independent of the amplitude.



a) Parametrize the shape of the surface by $x(\theta)$ and $y(\theta)$, as indicated in the figure. Write the Lagrangian for the pendulum.

b) What property must the Lagrangian have in order for the period of oscillation to be independent of the amplitude? Find the required shape $(x(\theta), y(\theta))$.

First, let's take a moment to recognize that Huygens solved this problem before Lagrangian mechanics was invented. He would have had to use results on the evolutes of cycloids, and stuff like that which no one cares about anymore. Still, very badass of him.

Anyway, let's solve it in a modern way. We'll clearly use θ as our coordinate. The position of the mass at angle θ is

$$(x(\theta) - \ell(\theta)\sin\theta, y(\theta) - \ell(\theta)\cos\theta)\,. \tag{1.1.19}$$

From this we can write the Lagrangian,

$$\begin{aligned} L = &\frac{1}{2}m\dot{\theta}^2\left(x'^2 + y'^2 + \ell^2 + \ell'^2 - 2\ell(x'\cos\theta - y'\sin\theta) - 2\ell'(x'\sin\theta + y'\cos\theta)\right) \\ &- mg(y(\theta) - \ell(\theta)\cos\theta). \end{aligned} \tag{1.1.20}$$

This is an unholy mess. Luckily we have a few constraints on ℓ, x, and y. The first is obvious: θ is the angle tangent to the curve, so

$$x' = y'\tan\theta. \tag{1.1.21}$$

The second comes from writing

$$\ell(\theta) = \ell_0 - \int_0^\theta \sqrt{x'(\theta)^2 + y'(\theta)^2}\,d\theta, \tag{1.1.22}$$

from which it follows that $\ell' = -\sqrt{x'^2 + y'^2}$. This simplifies the Lagrangian all the way down to

$$L = \frac{1}{2}m(\ell\dot{\theta})^2 - mg(y - \ell\cos\theta). \tag{1.1.23}$$

Now we try to cast this into the form of the Lagrangian for a harmonic oscillator, and see what constraint this imposes. Let

$$\frac{d\psi}{dt} = \ell\dot{\theta}, \tag{1.1.24}$$

so $\frac{d\psi}{d\theta} = \ell$. We then need

$$\psi^2 \propto y - \ell\cos\theta. \tag{1.1.25}$$

Differentiating with respect to θ gives

$$2\psi\ell \propto y' - \ell'\cos\theta + \ell\sin\theta = \ell\sin\theta, \tag{1.1.26}$$

where we use the constraints written above to simplify. Taking a time derivative, this implies $\ell = k\cos\theta$, and a θ derivative of this gives

$$y' = -k\sin(2\theta) \implies y = -\frac{k}{2}(1 - \cos(2\theta)). \tag{1.1.27}$$

Working from $x' = y'\tan\theta$, we then have

$$x = -\frac{k}{2}\left(2\theta - \sin(2\theta)\right). \tag{1.1.28}$$

These are the equations of a cycloid of radius $\frac{k}{2}$.

Finally, there is the Hamiltonian formulation of mechanics. The Hamiltonian formulation is brilliant and of fundamental importance to theoretical dynamics, quantization, and statistical mechanics. It is not, however, an indispensible tool for problem solving; typically the Lagrangian approach is just as fruitful. As a compromise, we'll look briefly at the formulation and use it to solve a problem that could also be solved with a Lagrangian.

The Hamiltonian formulation replaces the velocities $\dot{x}$ with the canonical momenta $p_x \equiv \frac{\partial L}{\partial \dot{x}}$, via the Legendre transformation

$$H(x, p) = \sum_x p_x\dot{x} - L. \tag{1.1.29}$$

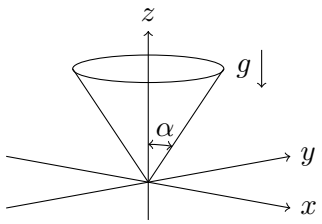
The dependence on $\dot{x}$ is replaced by dependence on p_x. In mechanics, we have $H = T + V$. The Euler-Lagrange equations imply the Hamilton principal equations,

$$\dot{x} = \frac{\partial H}{\partial p}, \qquad \dot{p} = -\frac{\partial H}{\partial x}. \tag{1.1.30}$$

Note that this is a first order system.

Problem 1.5 (M02M1)
A small particle of mass m is constrained to slide, without friction, on the inside of a circular cone whose vertex is at the origin and whose axis is along the z-axis. The half angle at the apex

of the cone is α and there is a uniform gravitational field g, directed downward and parallel to the axis of the cone.



a) Determine a set of generalized coordinates, and obtain the equations of motion in these coordinates.

b) Show that a solution of the equations of motion is a circular orbit at a fixed height z_0. Obtain an expression for the frequency, ω, of this motion.

c) Show that the circular motion is stable. If Ω is the frequency of small oscillations about the unperturbed motion, show that the ratio Ω/ω depends only on α. Determine this dependence.

We can use $r = \sqrt{x^2 + y^2}$ and the azimuthal angle θ as generalized coordinates. The Lagrangian is

$$L = \frac{1}{2}m\left(\dot{r}^2 \csc^2\alpha + r^2\dot{\theta}^2\right) - mgr\cot\alpha. \tag{1.1.31}$$

This gives canonical momenta

$$p_r = m\dot{r}\csc^2\alpha, \qquad p_\theta = mr^2\dot{\theta}. \tag{1.1.32}$$

The Hamiltonian is then

$$H = \frac{1}{2m}\left(p_r^2 \sin^2\alpha + \frac{p_\theta^2}{r^2}\right) + mgr\cot\alpha. \tag{1.1.33}$$

The Hamilton equations are

$$\dot{r} = \frac{p_r}{m}\sin^2\alpha, \qquad \dot{p}_r = \frac{p_\theta^2}{mr^3} - mg\cot\alpha, \tag{1.1.34}$$

$$\dot{\theta} = \frac{p_\theta}{mr^2}, \qquad \dot{p}_\theta = 0. \tag{1.1.35}$$

We obtain a circular orbit if $\dot{p}_r = 0$. The frequency of the orbit is given by $\omega = \frac{p_\theta}{mr_0^2}$, so we have

$$mr_0\omega^2 - mg\cot\alpha = 0 \implies \omega = \sqrt{\frac{g\cot\alpha}{r_0}} = \sqrt{\frac{g}{z_0}}\cot\alpha. \tag{1.1.36}$$

Now we perturb the orbit. It follows from the equations of motion that

$$\ddot{r} = \frac{\dot{p}_r}{m}\sin^2\alpha = \frac{p_\theta^2}{m^2r^3}\sin^2\alpha - mg\sin\alpha\cos\alpha. \tag{1.1.37}$$

The right hand side vanishes at $r = r_0$, so to first order in $u \equiv r - r_0$ we have

$$\ddot{u} + 3\frac{u}{m^2 r_0^4} p_\theta^2 \sin^2\alpha = 0. \tag{1.1.38}$$

It follows that

$$\Omega = \sqrt{3}\frac{p_\theta}{m r_0^2}\sin\alpha = \sqrt{3}\omega\sin\alpha. \tag{1.1.39}$$

1.2 Oscillators and Normal Modes

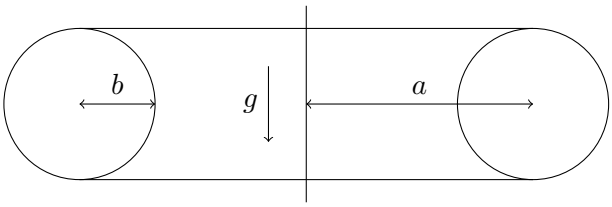
In Problem 1.5, we looked at small oscillations around a stable point in phase space, and found that the resulting equation of motion looks like that of a harmonic oscillator. This is a very general phenomenon. For a particle on a line, if the potential $V(x)$ has a critical point at $x = x_0$ with nonzero second derivative, then the equation of motion for a particle near that point is

$$m\ddot{x} + \left.\frac{d^2V}{dx^2}\right|_{x=x_0}(x - x_0) = 0. \tag{1.2.1}$$

This equation describes a harmonic oscillator with frequency

$$\omega = \sqrt{\frac{1}{m}\left.\frac{d^2V}{dx^2}\right|_{x=x_0}}. \tag{1.2.2}$$

Problem 1.6 (J01M1)
Find the frequency of small oscillations about uniform circular motion of a point mass that is constrained to move on the surface of a torus (donut) of major radius a and minor radius b whose axis is vertical.



Let θ be the angle around the inner circle of radius b, with $\theta = 0$ the bottom of the tube, and let ϕ be the angle around the outer circle of radius a. Then

$$L = \frac{1}{2}m\left((a + b\sin\theta)^2\dot{\phi}^2 + b^2\dot{\theta}^2\right) + mgb\cos\theta. \tag{1.2.3}$$

Clearly $\ddot{\phi} = 0$, and

$$b\ddot{\theta} = -g\sin\theta + (a + b\sin\theta)\cos\theta\dot{\phi}^2. \tag{1.2.4}$$

Setting the left hand side to zero, we find a circular orbit with

$$\dot{\phi} = \left(\frac{g \tan\theta_0}{a + b\sin\theta_0} \right)^{1/2}. \tag{1.2.5}$$

Now we perturb about this equilibrium. We have chosen $\dot{\phi}$ such that the right hand side (1.2.4) vanishes at θ_0, so

$$b\ddot{\theta} \approx \left(-g\cos\theta_0 + b\cos^2\theta_0 \dot{\phi}^2 - (a + b\sin\theta_0)\sin\theta_0 \dot{\phi}^2 \right)(\theta - \theta_0) \tag{1.2.6}$$

for $\theta \approx \theta_0$. Substituting for $\dot{\phi}$, we find

$$\ddot{\theta} + \left(\frac{g}{b}\sec\theta_0 - \frac{g\sin\theta_0\cos\theta_0}{a + b\sin\theta_0} \right)(\theta - \theta_0) = 0. \tag{1.2.7}$$

It is simple to show that the quantity in parentheses is positive, so the oscillations are stable with frequency

$$\Omega = \sqrt{\frac{g}{b}\sec\theta_0 - \frac{g\sin\theta_0\cos\theta_0}{a + b\sin\theta_0}}. \tag{1.2.8}$$

When there is more than one degree of freedom, the same sort of point holds, but there's a bit more work to be done. For a system with N degrees of freedom, the Lagrangian is generically of the form

$$L = \frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j - V(x^1, ..., x^n), \tag{1.2.9}$$

where g_{ij} is a constant symmetric tensor and repeated indices are summed. At a critical point $(x_0^1, \ldots, x_0^n)$ of V, all its first derivatives vanish and so we can locally express it as

$$V = V_0 + \frac{1}{2}\frac{\partial^2 V}{\partial x^i \partial x^j}(x^i - x_0^i)(x^j - x_0^j). \tag{1.2.10}$$

Substituting this into the Lagrangian, we obtain equations of motion

$$g_{ij}\ddot{x}^j + \frac{\partial^2 V}{\partial x^i \partial x^j}(x^j - x_0^j) = 0. \tag{1.2.11}$$

This looks like harmonic oscillator equations, but all the degrees of freedom are tangled up together, which is no good. To fix this, take the inverse matrix g^{ij} (i.e., such that $g^{ij}g_{jk} = \delta^i_k$, and multiply by it, giving

$$\ddot{x}^i + g^{ij}\frac{\partial^2 V}{\partial x^j \partial x^k}(x^k - x_0^k) = 0. \tag{1.2.12}$$

Now the accelerations are straightened out, but we have the matrix

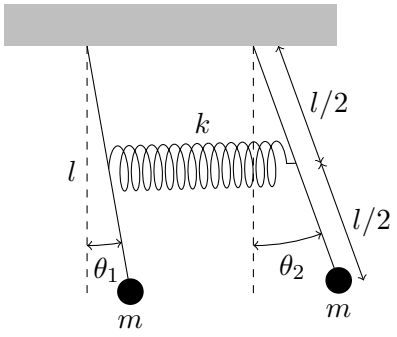
$$A^i_k \equiv g^{ij}\frac{\partial^2 V}{\partial x^j \partial x^k} \tag{1.2.13}$$

mixing up all the positions. The only way to avoid the mixing-up is to find the eigenvectors of A^i_k, and change coordinates so that these are the degrees of freedom.

The eigenvectors of A^i_k are called the normal modes of the system. If a system is perturbed along a normal mode, it will oscillate only in that mode, by construction. The eigenvalue corresponding

to a normal mode is the square of the frequency of that mode. It follows that the signature of A^i_k determines the stability of the critical point of V.

Problem 1.7 (J09M1)
Two simple pendula, each of length l and mass m, are coupled by a spring of force constant k. The spring is attached to the rods of the pendula, which are massless and inextensible, at their half-way points, as shown. Throughout, assume the angles θ_1 and θ_2 are small and that motion is confined to the 2D plane.



a) What are the normal frequencies of the system, and the corresponding normal mode vectors?

b) Consider now the case of "weak coupling" – i.e., the case when k is small. With respect to what is k small?

At $t = 0$ the lefthand pendulum is displaced by an angle $\theta_1(0) = \theta_0$ and released from rest; the righthand pendulum is at rest with $\theta_2 = 0$. Find expressions for $\theta_1(t)$ and $\theta_2(t)$ for $t > 0$. How long will it take before the lefthand mass stop swinging and the righthand mass achieves maximum amplitude?

The Lagrangian for the system is

$$L = \frac{ml^2}{2}(\dot\theta_1^2 + \dot\theta_2^2) - \frac{mgl}{2}(\theta_1^2 + \theta_2^2) - \frac{kl^2}{8}(\theta_2 - \theta_1)^2. \tag{1.2.14}$$

This gives equations of motion

$$\ddot\theta_1 + \frac{g}{l}\theta_1 + \frac{k}{4m}(\theta_1 - \theta_2) = 0, \tag{1.2.15}$$

$$\ddot\theta_2 + \frac{g}{l}\theta_2 + \frac{k}{4m}(\theta_2 - \theta_1) = 0. \tag{1.2.16}$$

Everything tangled up, as promised. As a matrix equation, this is

$$\begin{pmatrix}\ddot\theta_1\\ \ddot\theta_2\end{pmatrix} + \begin{pmatrix}\alpha & -\beta\\ -\beta & \alpha\end{pmatrix}\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix} = 0, \tag{1.2.17}$$

where $\alpha = \frac{k}{4m} + \frac{g}{l}$ and $\beta = \frac{k}{4m}$. The eigenvalues of the matrix are $\lambda_\pm = \alpha \pm \beta$, and so the normal mode frequencies are

$$\omega_+ = \sqrt{\frac{k}{2m} + \frac{g}{l}}, \qquad \omega_- = \sqrt{\frac{g}{l}}. \tag{1.2.18}$$

The normal modes are the eigenvectors $v_+ = (1, -1)$ and $v_- = (1, 1)$.

The system begins at displacement $(\theta_0, 0) = \frac{\theta_0}{2}(v_+ + v_-)$. Since we begin from rest, the displacement at time t will be given by

$$\frac{\theta_0}{2}\left(v_+ \cos\omega_+ t + v_- \cos\omega_- t\right). \tag{1.2.19}$$

We are looking for the first time when the displacement becomes $(0, c)$, which will occur when $\cos\omega_+ t = -\cos\omega_- t$. Assuming weak coupling ($kl \ll mg$), the frequencies are related by

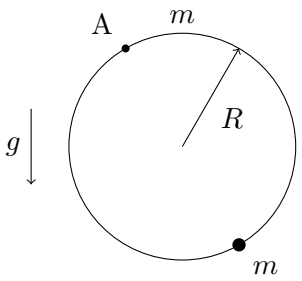
$$\omega_+ = \omega_- \sqrt{1 + \frac{kl}{2mg}} \approx \omega_- + \frac{k}{4m}\sqrt{\frac{l}{g}}. \tag{1.2.20}$$

At

$$t = 4\pi\sqrt{\frac{g}{l}}\frac{m}{k}, \tag{1.2.21}$$

we will have $\omega_+ t \approx \omega_- t + \pi$, so our condition is satisfied.

Problem 1.8 (J15M2)
A thin hoop of mass m and radius R is suspended from its rim (point A) and is free to rotate around point A in the plane of the hoop. A small bead of equal mass m can slide without friction on the hoop.



Find the frequencies of the normal modes for this system.

Let θ_h and θ_b be the angle of the hoop and bead respectively, both with respect to vertical. The center of mass positions are

$$x_h = (R\sin\theta_h, -R\cos\theta_h), \qquad x_b = (R(\sin\theta_h + \sin\theta_b), -R(\cos\theta_h + \cos\theta_b)). \tag{1.2.22}$$

Assuming small angles, this is

$$x_h = (R\theta_h, \frac{1}{2}R\theta_h^2), \qquad x_b = (R(\theta_h + \theta_b), \frac{1}{2}R(\theta_h^2 + \theta_b^2)). \tag{1.2.23}$$

The Lagrangian is then

$$L = \frac{1}{2}mR^2\left(2\dot{\theta}_h^2 + 2\dot{\theta}_h\dot{\theta}_b + \dot{\theta}_b^2\right) - \frac{1}{2}mgR\left(2\theta_h^2 + \theta_b^2\right). \tag{1.2.24}$$

This is of the form $\frac{1}{2}g_{ij}\dot{x}^i\dot{x}_j - \frac{1}{2}H_{ij}x^ix^j$, with

$$g_{ij} = mR^2\begin{pmatrix}2 & 1\\ 1 & 1\end{pmatrix}, \qquad H_{ij} = mgR\begin{pmatrix}2 & 0\\ 0 & 1\end{pmatrix}. \tag{1.2.25}$$

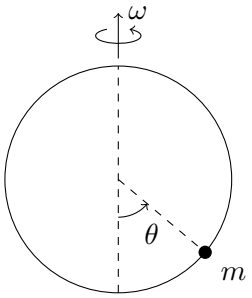
The matrix of interest is $A^i_k = g^{ij}H_{jk}$. Working this out, we find

$$A = \frac{g}{R}\begin{pmatrix}2 & -1\\ -1 & 2\end{pmatrix}. \tag{1.2.26}$$

The eigenvalues are $\frac{g}{R}$ and $3\frac{g}{R}$, so the normal mode frequencies are $\sqrt{\frac{g}{R}}$ and $\sqrt{3\frac{g}{R}}$.

The key property of simple harmonic oscillation is that the frequency is independent of the amplitude. If the system is at a degenerate point in phase space, small oscillations may be anharmonic. It's hard to come by clean analytic solutions for anharmonic oscillators.

Problem 1.9 (M00M3)
A circular hoop of radius a rotates about a vertical diameter with constant angular velocity ω. A small bead of mass m is constrained to slide without friction on the hoop. Consider the case when $\omega^2 = g/a$. The bead can undergo small oscillations around $\theta = 0$. These are not simple harmonic oscillations! Determine the period of these small oscillations. You may leave an unevaluated definite integral in your expression, but your solution should make it obvious how the period depends on the amplitude of oscillation.



The Lagrangian is

$$L = \frac{1}{2}m\left(a^2\omega^2\sin^2\theta + a^2\dot{\theta}^2\right) + mga\cos\theta, \tag{1.2.27}$$

so the equation of motion is

$$ma^2\ddot{\theta} + mga\sin\theta - ma^2\omega^2\sin\theta\cos\theta = 0. \tag{1.2.28}$$

For generic ω, there will be a term linear in θ, and the oscillation is harmonic. But since $\omega^2 = g/a$, the linear term cancels, so we are left with

$$\ddot{\theta} + \frac{g}{2a}\theta^3 = 0. \tag{1.2.29}$$

Our goal is to compute the time required for the bead to move from $-\theta_0$ to θ_0 and back again, or

$$T = \int dT = 4\int_0^{\theta_0} \frac{d\theta}{\dot{\theta}}. \tag{1.2.30}$$

We can determine $\dot{\theta}$ from the conservation of energy. Multiplying the equation of motion by $\dot{\theta}$, we find the first integral

$$\frac{d}{dt}\left(\dot{\theta}^2 + \frac{g}{4a}\theta^4\right) = 0, \tag{1.2.31}$$

so at angle θ, we have

$$\dot{\theta}^2 = \frac{g}{4a}(\theta_0^4 - \theta^4). \tag{1.2.32}$$

Substituting this into the integral, we find

$$T = 8\sqrt{\frac{a}{g}}\int_0^{\theta_0} \frac{d\theta}{\sqrt{\theta_0^4 - \theta^4}}. \tag{1.2.33}$$

To make the dependence on θ_0 explicit, define $u = \theta/\theta_0$, so

$$T = 8\sqrt{\frac{a}{g}}\frac{1}{\theta_0}\int_0^1 \frac{du}{\sqrt{1-u^4}}. \tag{1.2.34}$$

Problem 1.10 (M01M2)
A particle of mass m moves in a one-dimensional potential $V(x) = -ax^2 + bx^4$ with very light damping. The particle is set in motion with a large initial velocity. Suppose now we measure the period of the motion for each full oscillation, and call these periods T_1, T_2, T_3, T_4, and so on. It is observed that the T_i briefly become very large for i near some i_0.

a) Explain what makes the periods get large.

b) Obtain a scaling form for T_i near $i = i_0$, valid in the limit of small damping. (A scaling form would be something like $T \sim |i - i_0|^\alpha$ for some α, or $T \sim \log|i - i_0|$, etc). Hint: consider first the motion without the friction, $m\ddot{x} = -V'(x)$. Recalling that this motion is necessarily periodic, derive an integral formula relating the period of oscillation to the energy and the turning points x_- and x_+ of the motion.

c) Give an approximate sketch of T_i as a function of i.

It is clear why the periods get large. The potential has the form shown in Figure 1.4. As the energy gets close to zero, the particle will move very slowly over the central hump in the potential, making the period very large.

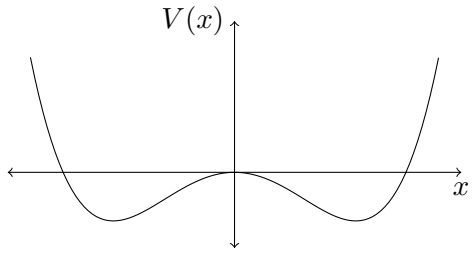


Figure 1.4

To address this quantitatively, we need to determine the period of a particle with energy E. We start by assuming $E > 0$, so this is given by

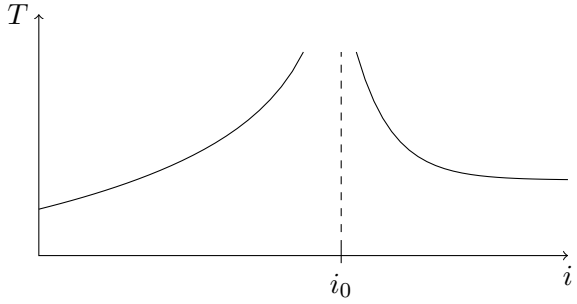
$$T = \sqrt{2m} \int_{x_-}^{x_+} \frac{dx}{\sqrt{E - V(x)}}, \tag{1.2.35}$$

where $x_- < x_+$ are the unique points at which $E = V(x)$. We need to determine the dominating contribution to the integral from the region where $|x|$ is small. In particular, we can take $|x| \ll \sqrt{a/b}$, so that the bx^4 term in the potential is negligible. We then have

$$T \approx \sqrt{2m} \int_{-\epsilon}^{\epsilon} \frac{dx}{\sqrt{E + ax^2}} = \sqrt{\frac{2m}{E}} \frac{1}{i\sqrt{a/E}} \sin^{-1}\left(i\sqrt{\frac{a}{E}}x\right)\Bigg|_{-\epsilon}^{\epsilon}. \tag{1.2.36}$$

Since we are taking E very small while the scale of ϵ is fixed, the argument of $\sin^{-1}$ here is large and imaginary. Looking at $\sin x = \frac{1}{2i}(e^{ix} - e^{-ix})$, it's clear that in order to get a large imaginary result we want a large imaginary argument. Working through the details, $\sin^{-1}(iy) \approx i \log(2y)$ for $|y| \gg 1$. Using this, we find that T scales as a logarithm of E. Since trajectories near the critical E all look similar, the damping should subtract a constant δE for each oscillation, so we have $T \sim \log|i - i_0|$.

Following this logarithmic divergence, the particle will be stuck on one side of the hump, and so its oscillation will be roughly harmonic. The period thus settles down to some constant value. The following sketch roughly captures the behavior.



1.3 Orbits and Scattering

Do you ever feel alone, adrift in an empty universe? The world is a cruel place. Sometimes it feels like no one wants you. We are condemned to live out our lives treading the soil of our pale blue dot in the sky, wondering if someone, somewhere in this vast cosmos, someone wants us.

In these dark times, it's important to remember that we are never alone. The gravitational field is all around us, and no matter how much emotional baggage you may be carrying, it will always pull. The heavier your problems, the harder it tugs.

This most reliable of companions can be described in a few equations. The force of gravity between two masses is

$$\boldsymbol{F} = -\frac{Gm_1m_2}{r^2}\hat{\boldsymbol{r}}, \tag{1.3.1}$$

where $\boldsymbol{r}$ is a displacement vector from one object to the other and the minus sign indicates that the force is always attractive. Since the inertial mass in $F = ma$ is the same as the gravitational mass appearing here - a fact of enormous importance - the mass cancels, and we can write

$$\ddot{\boldsymbol{x}} = -\nabla\phi, \tag{1.3.2}$$

where

$$\phi = \sum_i -\frac{Gm_i}{r_i}. \tag{1.3.3}$$

The scalar ϕ is called the gravitational potential, and its negative gradient is the gravitational field. Since $\nabla^2\left(\frac{1}{r}\right) = -4\pi\delta(x)$, we can also write this equation as

$$\nabla^2\phi = 4\pi G\rho, \tag{1.3.4}$$

where ρ is the mass density.

Problem 1.11 (J99M3)
A rod of length L, mass m and uniform mass density is circling in an orbit around the Sun at distance R (between the center of the rod and the Sun). The mass of the Sun is M. Assume that the rod is always pointed in the radial direction. Calculate the tension at the center of the rod.

The force of gravity on the rod is

$$F = -\frac{GMm}{R^2}\hat{\boldsymbol{r}}, \tag{1.3.5}$$

which must be equal in magnitude to the centripetal force $mR\omega^2$ if the rod is to maintain a circular trajectory. It follows that $\omega = \sqrt{\frac{GM}{R^3}}$.

In order to maintain rigidity, there must be some tension in the rod which pulls on the outer bits and pushes on the inner bits, so that they all experience the right amount of centripetal force. A piece of the rod of length dx at position x (measured from the center) will feel a force due to tension of $T(x+dx) - T(x)$, and it has a mass $\frac{m}{L}dx$. It needs a compensating inward force of

$\frac{m}{L}dx \times 3x\omega^2$ (the factor of 3 comes from 1 from the change in the centripetal force at $R+x$ and 2 from the change in the gravitational force). Therefore,

$$\frac{L}{m}T'(x) = -3x\omega^2. \tag{1.3.6}$$

Imposing boundary conditions $T(-L/2) = T(L/2) = 0$, this gives

$$T(x) = \frac{3GMmL}{8R^3} - \frac{3GMm}{2LR^3}x^2, \tag{1.3.7}$$

so in particular, $T(0) = \frac{3GMmL}{8R^3}$.

So, you're probably asking: just hypothetically, if no one else wants me and I'm desperate enough to strike up a relationship with an invisible field defined solely for calculation purposes, how might that relationship go? Johannes Kepler addressed this question during an unhappy marriage, and it was answered in fuller detail later on by the lifelong bachelor and the J. Crew model himself, Isaac Newton. The behavior of a test mass (e.g., a planet) in the gravitational field of a fixed point mass (e.g., a star) is conventionally described by Kepler's laws:

1. Planets orbit in conic sections (circles, ellipses, parabolae, or hyperbolae) with one of the foci at the star.
2. Planets sweep out equal areas in equal times.
3. For bound orbits, the period scales as $a^{3/2}$, where a is the semimajor axis.

The second law is trivial to prove; it expresses the conservation of angular momentum. We have

$$\frac{dA}{dt} = \frac{1}{2\,dt}(\boldsymbol{r} \times d\boldsymbol{r}) = \frac{\boldsymbol{L}}{2m}. \tag{1.3.8}$$

Angular momentum is conserved about the center of any central force (i.e., a force of the form $\boldsymbol{F}(\boldsymbol{r}) = f(r)\hat{\boldsymbol{r}}$), and the result follows.

The third law is easiest to see by looking at the scaling behavior of gravity. Consider again the equations of motion

$$\ddot{x} = -\nabla\phi, \qquad \nabla^2\phi = 4\pi G\rho. \tag{1.3.9}$$

It is straightforward to show that these equations are invariant under the rescaling

$$m \mapsto \alpha m, \qquad x \mapsto \beta x, \qquad t \mapsto \alpha^{-1/2}\beta^{3/2}t. \tag{1.3.10}$$

It follows that $t \propto a^{3/2}$.

The first law has some more meat on its bones. We won't prove it here, but we can sketch out how it works using the radial equation of motion. The Lagrangian for an orbiting planet is

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \frac{GMm}{r}, \tag{1.3.11}$$

which gives $mr^2\dot{\theta} = \text{const}$ and

$$m\ddot{r} - mr\dot{\theta}^2 + \frac{GMm}{r^2} = 0. \tag{1.3.12}$$

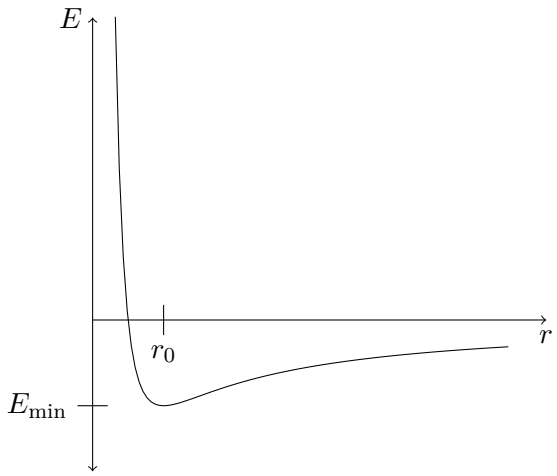


Figure 1.5: The effective potential sets the dynamics of the radial coordinate r.

Letting $L = mr^2\dot{\theta}$ (just to be deliberately confusing), this becomes

$$\ddot{r} + \frac{GMm}{r^2} - \frac{L^2}{mr^3} = 0. \tag{1.3.13}$$

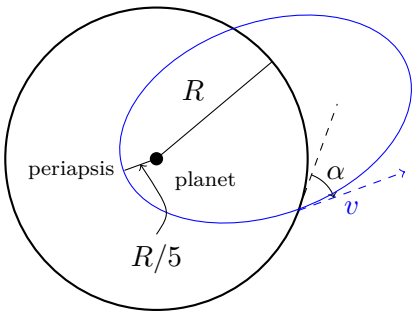
This looks like the equation we would get for a one-dimensional system with effective potential

$$V_{\text{eff}}(r) = -\frac{GMm}{r} + \frac{L^2}{2mr^2}. \tag{1.3.14}$$

Unless we're on a radial infall orbit, for which $L = 0$, the effective potential goes to infinity as $r \to 0$. This is sometimes called the centrifugal barrier. The effective potential takes the form shown in Figure 1.5, which leads to four possibilities for the energy:

- If $E = E_{\min}$, then the orbit is stuck at r_0. This is clearly a circular orbit.
- If $E_{\min} < E < 0$, then r oscillates between two values. This is an elliptical orbit.
- If $E = 0$, then r can escape to infinity, but with vanishingly small velocity. This turns out to be a parabolic orbit.
- If $E > 0$, then r can escape to infinity with finite velocity. This is a hyperbolic orbit.

Problem 1.12 (J14M1)
A satellite of mass m moves in a circular orbit of radius R about a much more massive planet (of unspecified mass). The satellite has speed v.



At a specific point in the satellite's circular orbit, the velocity of the satellite is abruptly rotated without changing the magnitude of its velocity. (The nature of this external impulse is not specified.) As shown in the figure, this causes the satellite to enter an elliptical orbit with its distance of closest approach = $R/5$. (This point in the elliptical orbit is called the periapsis in general.) The elliptical orbit is in the same plane as the circular orbit.

a) What is the speed v_p of the satellite at the periapsis in terms of v?

b) Through what angle α was the satellite turned? See figure for the definition of α.

Originally we have $\frac{mv^2}{R} = \frac{GMm}{R}$, so $v = \sqrt{\frac{GM}{R}}$. Since the magnitude of velocity is unchanged in the perturbation, the energy remains

$$E = -\frac{GMm}{R} + \frac{1}{2}mv^2 = -\frac{GMm}{2R}, \tag{1.3.15}$$

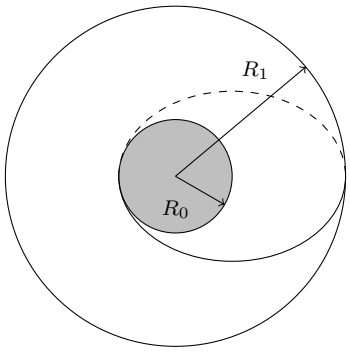
where M is the planet mass. Therefore, the velocity at periapsis is

$$v_p = \sqrt{\frac{2}{m}\left(\frac{5GMm}{R} - \frac{GMm}{2R}\right)} = 3\sqrt{\frac{GM}{R}} = 3v. \tag{1.3.16}$$

This implies that the angular momentum of the new orbit is $\frac{3}{5}$ that of the original orbit, so $\alpha = \cos^{-1}\frac{3}{5}$.

Problem 1.13 (M07M1)
A satellite in a low Earth circular orbit with Radius R_0 has an orbital period T_0.

a) How long does it take to transfer the satellite into a new circular orbit with a larger radius R_1 using the Hohmann transfer ellipse shown in the figure?



b) Suppose a large shower of asteroids (much larger than the Earth diameter) came to Earth from a distant source, all moving with the same initial velocity v. If the areal number density

of asteroids in the shower (the number of asteroids crossing a unit area perpendicular to the initial velocity) is n, how many of them will hit the Earth? You can ignore the effects of other bodies in the Solar system.

Part a) is a simple application of Kepler's third law. The semimajor axis of the Hohmann ellipse is $\frac{1}{2}(R_0 + R_1)$, so the half-period will be

$$T_{\text{transfer}} = \frac{1}{2} T_0 \left(\frac{R_0 + R_1}{2R_0} \right)^{3/2}. \tag{1.3.17}$$

In part b), we need to determine the periapsis of a mass with velocity v and impact parameter b. The energy of the orbit is clearly $\frac{1}{2}mv^2$ (where m is the asteroid mass), and the angular momentum is mvb. If the asteroid makes its closest approach at distance d with velocity $v_{\max}$, the conservation laws require

$$\frac{1}{2}mv_{\max}^2 - \frac{GMm}{d} = \frac{1}{2}mv^2, \qquad mv_{\max}d = mvb. \tag{1.3.18}$$

Solving the system of equations, we find

$$d = \sqrt{\frac{G^2M^2}{v^4} + b^2} - \frac{GM}{v^2}. \tag{1.3.19}$$

The asteroid will collide with Earth if $d < R$ (where R is the Earth radius). This gives

$$b^2 < \left(R + \frac{GM}{v^2} \right)^2 - \frac{G^2M^2}{v^4} = R^2 + \frac{2GMR}{v^2}. \tag{1.3.20}$$

All asteroids in this circle will impact, so we will get royally fucked over by

$$N = \pi n \left(R^2 + \frac{2GMR}{v^2} \right) \tag{1.3.21}$$

flying spheres of doom.

Gravity happens to have a $\frac{1}{r}$ potential, but we could construct a central force with any spherically symmetric $V(r)$ potential. Some of the properties of gravitational orbits will still hold true in this case. Circular orbits are always possible, and Kepler's second law holds for any orbit, since it only relies on conservation of angular momentum. However, non-circular orbits will generically not be conic sections, and indeed, even bound orbits may not be closed. The Bertrand theorem says that, among power-law potentials, only for $V \propto r^{-1}$ and $V \propto r^2$ are all bound orbits closed.

Problem 1.14 (M15M2)
A particle of mass m moves under the influence of an attractive central force with potential $V(r)$.

a) Suppose the particle to move in a circular orbit with angular momentum ℓ. Derive the

condition that determines how the orbit radius r_ℓ depends on ℓ.

b) Now consider a small perturbation $\delta r(t)$ around such a circular orbit. What is the condition that must be met for this perturbation to oscillate with a real frequency ω_ℓ (in which case the perturbation will not grow with time and the circular orbit will be stable)?

c) Now consider the special case $V(r) = -k/r^n$ with $n > 0$. For what values of n and ℓ are circular orbits stable (i.e. such that $\omega_\ell^2 \geq 0$)?

d) Are there any circumstances where the period of the circular orbit matches the period of small radial oscillations about the circular orbit. If so, what does this equality imply for the long-time trajectory of the particle?

The effective potential is $V_{\text{eff}}(r) = \frac{\ell^2}{2mr^2} + V(r)$. In a circular orbit, r must be at a critical point of this potential, so

$$-\frac{\ell^2}{mr_\ell^3} + \left.\frac{dV}{dr}\right|_{r=r_\ell} = 0. \tag{1.3.22}$$

In order for the orbit to be stable, the critical point must be a minimum, so

$$\frac{d^2V_{\text{eff}}}{dr^2} = \frac{3\ell^2}{mr_\ell^4} + \left.\frac{d^2V}{dr^2}\right|_{r=r_\ell} > 0. \tag{1.3.23}$$

The frequency will be $\omega_\ell = \sqrt{\frac{3\ell^2}{mr_\ell^4} + \left.\frac{d^2V}{dr^2}\right|_{r=r_\ell}}$.

Substituting $V(r) = -kr^{-n}$, we find

$$\frac{3\ell^2}{mr_\ell^4} - \frac{kn(n+1)}{r_\ell^{n+2}} > 0. \tag{1.3.24}$$

We additionally have the criticality condition

$$-\frac{\ell^2}{mr^3} + \frac{kn}{r_\ell^{n+1}} = 0. \tag{1.3.25}$$

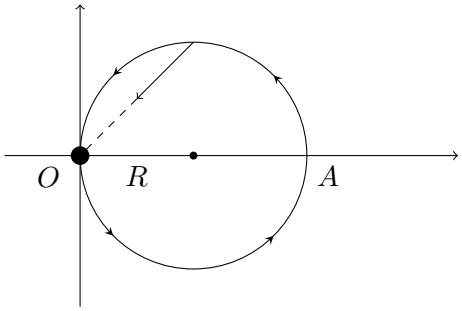
Using this to eliminate r_ℓ, we find that circular orbits are stable for any $n < 2$.

The frequency of oscillations is given by

$$\omega = \sqrt{2-n}\frac{\ell}{mr_\ell^2}, \tag{1.3.26}$$

and the frequency of the orbit is $\Omega = \frac{\ell}{mr_\ell^2}$. Clearly these are equal when $n = 1$, implying that for a gravitational potential $V(r) \propto r^{-1}$, bound orbits are closed.

Problem 1.15 (J06M2)
A point mass m is moving on a circular orbit of radius R under the effect of a central force directed toward the point O on the orbit (see figure below). Its speed at point A (A is diametrically opposite to O) is equal to v_A.



a) Find the expression for the force generating this motion.

b) Using the convention that the potential energy vanishes infinitely far from the center of attraction, compute the values of the energy and of the angular momentum for the circular orbit.

c) Find the time needed for the point mass to complete the orbit.

The angular momentum of the orbit is $L = 2mv_AR$, from which we can compute the velocity at each point on the circle. A bit of geometry shows that, if we measure θ with respect to the center of the circle with $\theta = 0$ at A, the distance to O is $2R\left|\cos\frac{\theta}{2}\right|$, and the angle between the velocity and the vector towards O is $\frac{\pi+\theta}{2}$. Thus,

$$2mv_AR = v(\theta) \times 2R\left|\cos\frac{\theta}{2}\right| \times \left|\cos\frac{\theta}{2}\right|, \tag{1.3.27}$$

or

$$v(\theta) = v_A \sec^2\frac{\theta}{2}. \tag{1.3.28}$$

Conservation of energy tells us that

$$\frac{1}{2}mv_A^2 \sec^4\frac{\theta}{2} + V\left(2R\left|\cos\frac{\theta}{2}\right|\right) = \frac{1}{2}mv_A^2 + V(2R). \tag{1.3.29}$$

If we require $V(\infty) = 0$, this gives

$$V(x) = -mv_A^2\frac{8R^4}{x^4}. \tag{1.3.30}$$

This implies the force is attractive with $F(r) = -\frac{32mv_A^2R^4}{x^5}$. The energy of the orbit vanishes.

The time needed to complete the orbit is

$$T = \int_0^{2\pi}\frac{d\theta}{v(\theta)/R} = \frac{2R}{v_A}\int_0^{\pi}\cos^2\frac{\theta}{2}\,d\theta = \frac{\pi R}{v_A}. \tag{1.3.31}$$

1.4 Rigid Bodies

A lot of things in the world are big and not squishy. It is worth developing a general framework for big things that aren't squishy. This is called rigid body dynamics.

Point particles have three degrees of freedom, the coordinates of their position. A rigid body has an additional three degrees of freedom specifying its orientation. In order to fully describe the dynamics of a rigid body, we need to determine the motion of its center of mass as well as changes in its orientation.

"A lot of problems are made easier by stabbing something" $\sim$ a serial killer. If we take a rigid body and skewer it with a fixed axis of rotation, then it has only one rotational degree of freedom. To determine how this degree of freedom responds to applied forces, we consider angular momentum, defined for a point particle by

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}, \tag{1.4.1}$$

where $\boldsymbol{r}$ is position relative to some fixed point. The angular momentum changes according to

$$\frac{d\boldsymbol{L}}{dt} = \boldsymbol{r} \times \boldsymbol{F}. \tag{1.4.2}$$

The quantity on the right is torque,

$$\boldsymbol{\tau} = \boldsymbol{r} \times \boldsymbol{F}. \tag{1.4.3}$$

Just as force is an exchange of momentum, torque is an exchange of angular momentum.

We are not concerned with points so much as extended bodies, so we should determine the angular momentum of a body with mass distribution $\rho(\boldsymbol{x})$ rotating about a fixed axis with angular velocity $\boldsymbol{\omega}$. The momentum of a differential mass $\rho(\boldsymbol{x})\,d\boldsymbol{x}$ will be $\boldsymbol{p} = \rho(\boldsymbol{x})\boldsymbol{\omega} \times \boldsymbol{r}\,d\boldsymbol{x}$, where $\boldsymbol{r}$ is measured from the axis, so

$$\boldsymbol{L} = \int \rho(\boldsymbol{x})\,(\boldsymbol{r} \times (\boldsymbol{\omega} \times \boldsymbol{r}))\;d\boldsymbol{x} = \left(\int \rho(\boldsymbol{x}) r^2\,d\boldsymbol{x}\right)\boldsymbol{\omega}. \tag{1.4.4}$$

The quantity in parentheses is the moment of inertia, I, relative to the axis specified by $\boldsymbol{\omega}$. We write this relation as

$$\boldsymbol{L} = I\boldsymbol{\omega}. \tag{1.4.5}$$

Combining this with (1.4.2), we find

$$\boldsymbol{\tau} = I\boldsymbol{\alpha}, \tag{1.4.6}$$

where $\boldsymbol{\alpha}$ is the angular acceleration.

Computing the moment of inertia of an object is mildly annoying. Here are some common objects and their moments of inertia.

- Uniform rod of mass m and length L about a perpendicular axis through its center of mass: $I = \frac{1}{12}mL^2$
- Uniform disk of mass m and radius r about a perpendicular axis through its center of mass: $I = \frac{1}{2}mr^2$
- Uniform ring of mass m and radius r about a perpendicular axis through its center of mass: $I = mr^2$

- Uniform filled sphere of mass m and radius r about a perpendicular axis through its center of mass: $I = \frac{2}{5}mr^2$
- Uniform hollow sphere of mass m and radius r about a perpendicular axis through its center of mass: $I = \frac{2}{3}mr^2$

Furthermore, if the axis is shifted by a distance d from the center of mass, the moment of inertia increases by md^2. This is simple to prove.

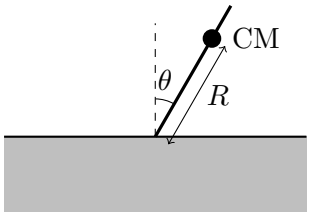
The conservation of energy will also hold for rigid bodies, provided we can write down an expression for the kinetic energy due to rotation. The squared velocity of an infinitesimal chunk of mass is $|\boldsymbol{\omega} \times \boldsymbol{r}|^2 = \omega^2 r^2$ (since $\boldsymbol{r}$ is perpendicular to the axis), so we find

$$T = \frac{1}{2}I\omega^2, \tag{1.4.7}$$

in addition to any kinetic energy due to linear motion.

Problem 1.16 (J04M3)
A thin stick with some arbitrary linear mass density $\mu(x)$ along it is initially at rest. It has one end on a table and makes an angle θ_0 with the vertical. The stick-table contact point has an infinite coefficient of friction.

Let m be the total mass of the stick, R be the distance from the contact point to the center of mass, I_{CM} be the moment about the center of mass, and g be the acceleration due to gravity.



a) The stick is released from rest and allowed to fall to the table. Find the condition that the end of the stick initially in contact with the table *does* rise from the table as the stick falls. Express the condition in terms of θ_0, m, g, R, and I_{CM}.

b) Now consider a specific mass distribution. Let the mass be uniformly distributed along the length. For what range of initial angles θ_0 will the stick eventually lift off the table?

c) Consider a different mass distribution: the mass is concentrated in two points of equal mass, one at either end of the stick. Now for what range of initial angles θ_0 will the stick eventually lift off the table?

The stick will lift off the table if the normal force at the contact point ever vanishes. We have three tools at our disposal with which to compute this normal force: conservation laws of energy, linear momentum, and angular momentum. Conservation of energy gives

$$\frac{1}{2}I_{CM}\dot{\theta}^2 + \frac{1}{2}mR^2\dot{\theta}^2 = mgR(\cos\theta_0 - \cos\theta). \tag{1.4.8}$$

Conservation of linear momentum in the vertical direction gives

$$m\frac{d^2}{dt^2}\left(R\cos\theta\right) = N - mg, \tag{1.4.9}$$

where N is the normal force, and conservation of angular momentum about the pivot point gives

$$(I_{CM} + mR^2)\ddot{\theta} = mgR\sin\theta. \tag{1.4.10}$$

Combining these three equations, we have

$$N = mg\left(1 - \frac{2}{1+\alpha}(\cos\theta_0 - \cos\theta)\cos\theta - \frac{1}{1+\alpha}\sin^2\theta\right), \tag{1.4.11}$$

where $\alpha = \frac{I_{CM}}{mR^2}$. Now we solve for $N = 0$. It's a quadratic equation in $\cos\theta$, which is why I totally didn't use Mathematica to tell me that

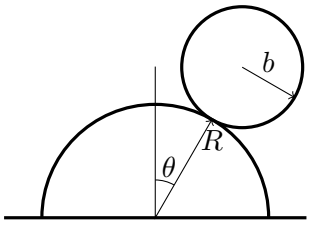
$$\cos\theta = \frac{1}{3}\left(\cos(\theta_0) - \sqrt{\cos^2\theta_0 - 3\alpha}\right). \tag{1.4.12}$$

Thus, our constraint is $\frac{3I_{CM}}{mR^2} < \cos^2\theta_0$.

If the mass is uniformly distributed, we have $I_{CM} = \frac{1}{3}mR^2$, so the stick will never lift off the table (though it will come oh so close as $\theta_0 \to 0$).

If the mass is concentrated at either end of the stick, we have $I_{CM} = mR^2$, so again the stick will never lift off the table.

Problem 1.17 (J15M3)
A uniform cylinder of mass m and radius b rolls off a fixed cylindrical surface of radius R under the influence of gravity. The axes of both cylinders are horizontal. The rolling cylinder starts from the top of the fixed cylinder with a negligibly small velocity.



a) If we assume the cylinder rolls without slipping, find the angle θ from the vertical when it loses contact with the fixed cylinder.

b) In practice for a finite value of μ the cylinder will start to slip before it loses contact. Find the angle when it starts to slip for $\mu = 1$.

We denote the angular velocity of the rolling cylinder by Ω. The no-slip condition is $b\Omega = R\dot{\theta}$, and the center of mass velocity of the rolling cylinder is $v = R\dot{\theta}$.

We need to determine when the normal force between the cylinders vanishes. The centripetal force on the rolling cylinder is

$$\frac{mv^2}{R+b} = mg\cos\theta - N. \tag{1.4.13}$$

Conservation of energy tells us that

$$\frac{1}{2}\left(\frac{1}{2}mb^2\right)\Omega^2 + \frac{1}{2}mv^2 = mg(R+b)(1-\cos\theta). \tag{1.4.14}$$

We solve and find

$$v^2 = \frac{4}{3}g(R+b)(1-\cos\theta). \tag{1.4.15}$$

Setting the normal force to zero, this implies

$$\cos\theta - \frac{4}{3}(1-\cos\theta) = 0. \tag{1.4.16}$$

The solution is

$$\theta = \cos^{-1}\frac{4}{7}. \tag{1.4.17}$$

When we turn on a frictional force F, we have

$$mR\ddot{\theta} = mg\sin\theta - F. \tag{1.4.18}$$

Furthermore, F provides the torque to rotate the cylinder, so $Fb = \frac{1}{2}mb^2\dot{\Omega}$. Therefore,

$$\dot{\Omega} = \frac{2}{3}\frac{g}{b}\sin\theta \tag{1.4.19}$$

and

$$F = \frac{1}{3}mg\sin\theta. \tag{1.4.20}$$

The cylinder will slip if this exceeds the maximal frictional force μN. With $\mu = 1$, this condition is

$$\frac{1}{3}mg\sin\theta > \left(mg\cos\theta - \frac{mv^2}{R+b}\right). \tag{1.4.21}$$

Substituting the velocity from energy conservation (since energy is conserved up until the cylinder slips), we find

$$7\cos\theta - \sin\theta < 4. \tag{1.4.22}$$

Solving this, we find that the cylinder first slips at

$$\theta = \cos^{-1}\frac{28+\sqrt{34}}{50}. \tag{1.4.23}$$

Nobody said it had to be pretty.

Serial killers are not always there to save us. Sometimes, rigid bodies haven't been stabbed or otherwise constrained, and they can rotate every which way. In this case, we find a more general relationship between angular momentum and angular velocity. Rather than a scalar moment of inertia, they are related by an inertia tensor

$$I_{ij} = \int (x^2 \delta_{ij} - x_i x_j) \rho(\boldsymbol{x})\, d\boldsymbol{x}, \tag{1.4.24}$$

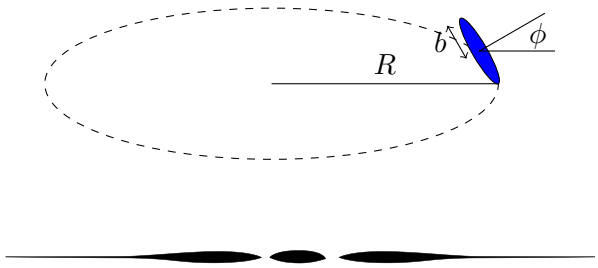
and the angular momentum is given by

$$L^i = I^i{}_j \omega^j. \tag{1.4.25}$$

Similarly, the kinetic energy is $T = \frac{1}{2} I_{ij} \omega^i \omega^j$.

The inertia tensor is symmetric, and so the spectral theorem implies that its eigenvectors form an orthonormal basis. The eigenvectors are called the principal axes of the rigid body.

Problem 1.18 (J11M2)
A coin (uniform solid cylinder) of mass M and radius b rolls without slipping on a horizontal table such that the axis perpendicular to its face makes a constant angle ϕ with respect to the table top (see diagram). The point of contact moves in a counterclockwise (as viewed from above) circular path of radius R with constant linear speed v. What is the relationship between ϕ and the given quantities? In your solution, do not assume that ϕ is a small angle.



We start by computing the angular momentum about the center of the circle formed by the center of mass of the disk as it precesses. The inertia tensor of the coin is given by

$$I_{ij} = \frac{1}{4} m b^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \tag{1.4.26}$$

where $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are principal axes in the plane of the disk and $\boldsymbol{e}_3$ is perpendicular to the disk. The spin angular velocity $\omega = \frac{v}{b}$ is directed along $-\boldsymbol{e}_3$, so the angular momentum due to spin is

$$\boldsymbol{L}_s = -\frac{1}{2} m v b \boldsymbol{e}_3. \tag{1.4.27}$$

The precession angular velocity $\Omega = \frac{v}{R}$ is directed along $\hat{\boldsymbol{z}} = \cos\phi \boldsymbol{e}_1 + \sin\phi \boldsymbol{e}_3$, so the angular momentum due to precession is

$$\boldsymbol{L}_p = \frac{1}{4} m b^2 \frac{v}{R} (\cos\phi \boldsymbol{e}_1 + 2 \sin\phi \boldsymbol{e}_3). \tag{1.4.28}$$

In the lab frame, $\boldsymbol{e}_1 = \cos\phi\hat{\boldsymbol{z}} - \sin\phi\hat{\boldsymbol{r}}$ and $\boldsymbol{e}_3 = \cos\phi\hat{\boldsymbol{r}} + \sin\phi\hat{\boldsymbol{z}}$. Using these identities, we find

$$\frac{\boldsymbol{L}_{\text{tot}}}{mvb} = \frac{1}{2}\left(-\sin\phi + \frac{b}{R}\cos(2\phi)\right)\hat{\boldsymbol{z}} + \frac{1}{2}\left(-\cos\phi + \frac{1}{4}\frac{b}{R}\sin(2\phi)\right)\hat{\boldsymbol{r}}. \tag{1.4.29}$$

Since $\frac{d\hat{\boldsymbol{r}}}{dt} = \Omega\hat{\boldsymbol{\theta}}$, we have

$$\frac{d\boldsymbol{L}}{dt} = \frac{mbv^2}{2R}\left(-\cos\phi + \frac{b}{4R}\sin(2\phi)\right)\hat{\boldsymbol{\theta}}. \tag{1.4.30}$$

We now need to set this equal to the torque. The torque comes from gravity acting on the center of mass of the disk, and the normal force and and friction acting on its edge. The magnitude of the gravitational and normal forces are obvious, and that of the frictional force is set by kinematics to $F_f = m(R - b\sin\phi)\Omega^2$. We have

$$\boldsymbol{\tau} = -mgb\sin\phi + \frac{mv^2}{R^2}(R - b\sin\phi)(b\cos\phi). \tag{1.4.31}$$

Setting this equal to $\frac{d\boldsymbol{L}}{dt}$, we end up with

$$\frac{3R}{2}\cot\phi - \frac{5b}{4}\cos\phi = \frac{gR^2}{v^2}. \tag{1.4.32}$$

If we were to assume small ϕ, the first term on the left would dominate and we would have $\tan\phi = \frac{3v^2}{2gR}$.

It was a bit of a pain to convert from the body frame back to the lab frame in the previous problem. In this case the pain was basically inevitable one way or another, but sometimes it's very useful to stay in the body frame. The difficulty is that the body frame is rotating with the body. So, if we want to write $\frac{d\boldsymbol{L}}{dt}$ using $\boldsymbol{L}$ in the body frame, we have to account for changes in $\boldsymbol{L}$ relative to the body as well as the rotation of the body itself.

Luckily there's a straightforward way to do this bookkeeping. If we have a vector $\boldsymbol{A}$ affixed to a coordinate system which rotates with angular velocity $\boldsymbol{\omega}$, it's straightforward to see that the time derivative of $\boldsymbol{A}$ due to frame rotation will be $\boldsymbol{\omega}\times\boldsymbol{A}$. If $\boldsymbol{A}$ is also changing in the rotating coordinate system, we simply add this change in. So, we have

$$\frac{d\boldsymbol{A}}{dt} = \left(\frac{d\boldsymbol{A}}{dt}\right)_{\text{rot}} + \boldsymbol{\omega}\times\boldsymbol{A}. \tag{1.4.33}$$

Applying this to the angular momentum, we have

$$\left(\frac{d\boldsymbol{L}}{dt}\right)_{\text{rot}} + \boldsymbol{\omega}\times\boldsymbol{L} = \boldsymbol{\tau}. \tag{1.4.34}$$

Since we are in a body-centered frame, we can use the principal axes, so that $L_i = I_i\omega_i$. Carrying out the cross product, we have the Euler equations,

$$\begin{aligned} I_1\dot{\omega}_1 + (I_2 - I_3)\omega_2\omega_3 &= \tau_1, \\ I_2\dot{\omega}_2 + (I_3 - I_1)\omega_1\omega_3 &= \tau_2, \\ I_3\dot{\omega}_3 + (I_1 - I_2)\omega_1\omega_2 &= \tau_3. \end{aligned} \tag{1.4.35}$$

Problem 1.19 (J01M3)
Suppose the object can be treated as a rigid body whose principal moments of inertia obey $(I_P - I_E)/I_E = \epsilon$ to deduce the angular frequency Ω of free precession in terms of the angular frequency ω of rotation.

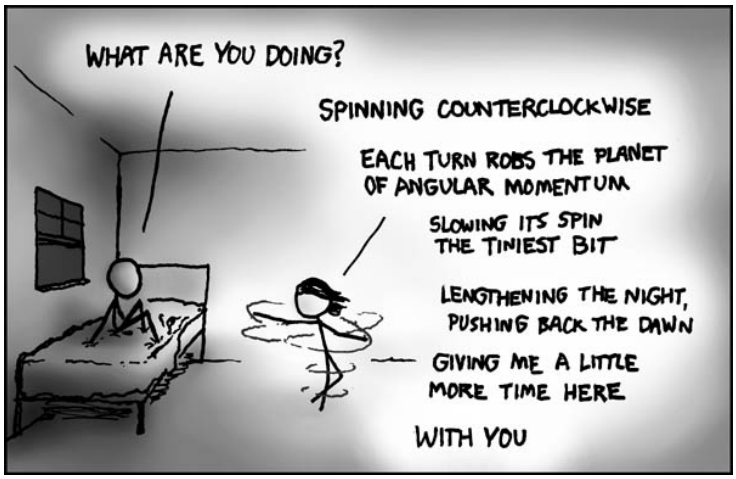


Figure 1.6: The leading-order correction to the zero torque assumption in this problem.

This problem is considering a planet with an equatorial bulge. We can take $I_1 = I_P$ and $I_2 = I_3 = I_E$, so $\omega_1 = \omega$. We assume there is no torque on the planet, Figure 1.6 notwithstanding. The first Euler equation tells us that $\dot{\omega} = 0$, and the remaining two become

$$\begin{aligned} I_E\dot{\omega}_2 + (I_E - I_P)\omega\omega_3 &= 0, \\ I_E\dot{\omega}_3 + (I_P - I_E)\omega\omega_2 &= 0. \end{aligned} \tag{1.4.36}$$

Differentiating the first equation and substituting the second, we find

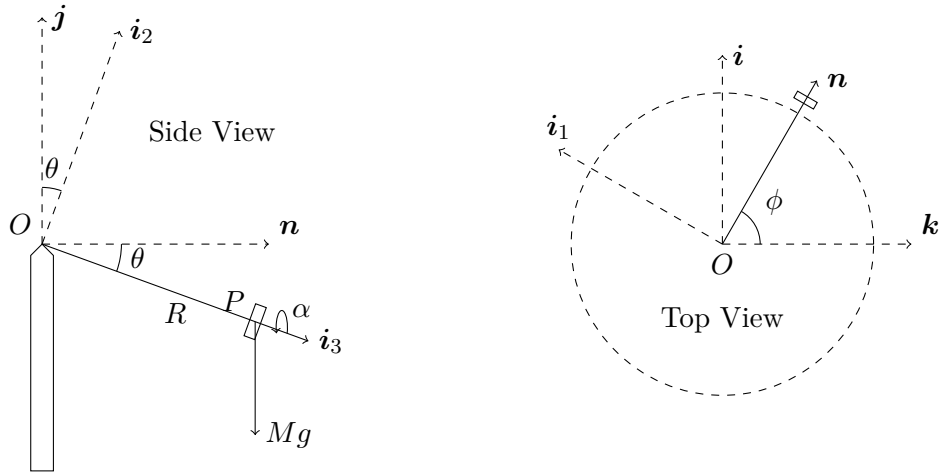
$$\ddot{\omega}_2 + \left(\frac{I_P - I_E}{I_E}\omega\right)^2 \omega_2 = 0. \tag{1.4.37}$$

Therefore, $\Omega = \omega\epsilon$. Easy peasy planet squeezy.

Of course, we can also take a Lagrangian approach to problems in rigid body dynamics. This is best illustrated by example.

Problem 1.20 (J06M1)
A gyroscope, illustrated in the figures below, is free to pivot about point O under the effect of gravity. Its total mass is M and its center of mass is located at point P at a distance R from O. In the reference frame $(O; \boldsymbol{i}_1, \boldsymbol{i}_2, \boldsymbol{i}_3)$, of the gyroscope (see figures), its moment of inertia tensor about point O is $\hat{I} = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I_3 \end{pmatrix}$. If $(O; \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$ is the laboratory frame and $\boldsymbol{n}$ that axis at

the intersection between the plane $\boldsymbol{i}_2\boldsymbol{i}_3$ and the plane $\boldsymbol{ik}$, define α to be the rotation angle of the gyroscope around $\boldsymbol{i}_3$, θ (the nutation angle) to be the angle between $\boldsymbol{i}_3$ and $\boldsymbol{n}$ and ϕ (the precession angle) as the angle between $\boldsymbol{k}$ and $\boldsymbol{n}$.



a) Write the Lagrangian of the system and its energy in terms of the angles α, θ, ϕ, and of their time derivatives.

b) Write the conservation laws for this system: energy and two projections of angular momentum.

c) From the conservation laws deduce a closed equation for θ in the form $F(\theta, \dot{\theta}) = 0$.

d) At time $t = 0$ the gyroscope is placed horizontally ($\theta = 0$) with zero nutation angular velocity ($\dot{\phi} = \dot{\theta} = 0$) and spin angular velocity $\dot{\alpha} = L_0/I_3$. Show that for $\theta \ll 1$ the previous equation and these initial conditions admit an approximate solution $\theta = \theta_0(1 - \cos\omega_n t)$. Compute the frequency ω_n, the amplitude θ_0, and the average precession velocity $\langle\dot{\phi}\rangle$. Find the condition on the initial data (i.e. on L_0) for which $\theta \ll 1$ remains a good approximation at all times.

We start by writing the angular velocity in the body frame as

$$\boldsymbol{\omega} = \dot{\alpha}\boldsymbol{i}_3 + \dot{\theta}\boldsymbol{i}_1 + \dot{\phi}(\cos\theta\boldsymbol{i}_2 - \sin\theta\boldsymbol{i}_3). \tag{1.4.38}$$

The kinetic energy is

$$T = \frac{1}{2}I_{ij}\omega^i\omega^j = \frac{1}{2}I(\dot{\theta}^2 + \cos^2\theta\dot{\phi}^2) + \frac{1}{2}I_3(\dot{\alpha} - \dot{\phi}\sin\theta)^2. \tag{1.4.39}$$

The potential energy is simply $-MgR\sin\theta$. The Lagrangian is then

$$L = \frac{1}{2}I(\dot{\theta}^2 + \cos^2\theta\dot{\phi}^2) + \frac{1}{2}I_3(\dot{\alpha} - \dot{\phi}\sin\theta)^2 + MgR\sin\theta. \tag{1.4.40}$$

The energy is obtained simply by flipping a sign,

$$E = \frac{1}{2}I(\dot{\theta}^2 + \cos^2\theta\dot{\phi}^2) + \frac{1}{2}I_3(\dot{\alpha} - \dot{\phi}\sin\theta)^2 - MgR\sin\theta. \tag{1.4.41}$$

The Lagrangian does not depend on α or ϕ, so their momenta are conserved:

$$L_\phi = I\cos^2\theta\dot{\phi} - I_3\sin\theta(\dot{\alpha} - \dot{\phi}\sin\theta), \qquad L_\alpha = I_3(\dot{\alpha} - \dot{\phi}\sin\theta). \tag{1.4.42}$$

Substituting these into the energy, we find

$$0 = \frac{1}{2}I\dot{\theta}^2 + \frac{(L_\phi + L_\alpha\sin\theta)^2}{2I} + \frac{L_\alpha^2}{2I_3} - MgR\sin\theta - E, \tag{1.4.43}$$

our desired equation for θ.

The initial conditions described correspond to an energy of $E = \frac{L_0^2}{2I_3}$ and angular momenta $L_\phi = 0$, $L_\alpha = L_0$, so we have

$$0 = \frac{1}{2}I\dot{\theta}^2 + \frac{L_0^2\sin^2\theta}{2I} - MgR\sin\theta. \tag{1.4.44}$$

If we take $\theta = \theta_0(1-\cos\omega_n t)$ and substitute while assuming small θ, we obtain

$$0 = \frac{1}{2}I\theta_0^2\omega_n^2\sin^2\omega_n t + \frac{L_0^2\theta_0^2}{2I}(1-\cos\omega_n t)^2 - MgR\theta_0(1-\cos\omega_n t). \tag{1.4.45}$$

If we set $\omega_n^2 = \frac{L_0^2}{I^2}$, this reduces to

$$0 = \frac{L_0^2\theta_0^2}{I}(1-\cos\omega_n t) - MgR\theta_0(1-\cos\omega_n t), \tag{1.4.46}$$

and we can satisfy this by fixing $\theta_0 = \frac{MgRI}{L_0^2}$. Thus, we need $L_0 \gg \sqrt{MgRI}$ to make the approximation valid. The average precession angular velocity is

$$\langle\dot{\phi}\rangle = \left\langle\frac{L_0\sin\theta}{I\cos^2\theta}\right\rangle = \frac{MgR}{L_0}. \tag{1.4.47}$$

1.5 Continuum Mechanics

The most general sorts of things are big and squishy. We will take "squishy" to include elastic solids, fluids, or really anything that doesn't fit into point particles or rigid bodies – this section is just a catch-all.

The simplest non-rigid bodies are springs. Springs have some natural length x_0, and a quadratic potential energy $\frac{1}{2}k(x-x_0)^2$ for displacements around that natural length. They serve as mechanical analogies for any sort of small oscillation about an equilibrium. That is, we basically covered the basics of springs in Sec. 1.2. It's worth noting that spring constants add in parallel and combine like capacitors in series.

Problem 1.21 (M06M3)
A spring has spring constant K, unstretched length L, and mass per unit length ρ (when unstretched). The spring is suspended from one end in a constant gravitational field, g, and

Figure 1.7: Be prepared for squishies by learning continuum mechanics. Otherwise you'll get stung.

stretches under its own weight. For a point whose distance from the upper end of the spring is x when unstretched, find its distance $y(x)$ from the upper end when the spring is stretched.

We use the fact, easy to prove, that spring constants combine like capacitors in series. This implies that if we take a portion x of the spring, it will have an effective spring constant $K\frac{L}{x}$. Thus, the spring will provide a restoring force

$$F = \frac{KL}{x}(y(x) - x) \tag{1.5.1}$$

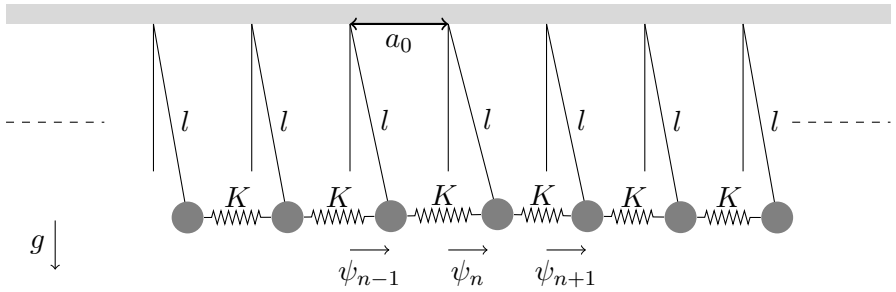
to the part of the spring below x. This should balance the gravitational force $\rho g\left(1 - \frac{x}{L}\right)$. This implies

$$y(x) = x + \frac{\rho g}{KL}x - \frac{\rho g}{KL^2}x^2. \tag{1.5.2}$$

Springs make a nice model for elastic media. Elastic media tend to support waves, i.e., phenomena which propagate through space over time. These waves will have a wavelength λ, or a wavenumber $k = 2\pi/\lambda$, which sets their behavior in space; and a frequency f, or angular frequency ω, which sets their behavior in time. The whole point of waves is that these things aren't independent, i.e., a given pattern in space will oscillate in a fixed way over time. This information is carried in the dispersion relation $\omega(k)$. For example, $\omega(k) = ck$ is the dispersion relation for a medium in which all waves propagate at speed c. If $\omega(k)$ becomes imaginary, disturbances decay rather than

propagating.

Problem 1.22 (J05M3)
Consider wave propagation in a one-dimensional medium which consists of a large number of pendula of mass m and length l coupled by springs of spring constant K. The distance between adjacent masses is a_0, which is also the natural length of the springs.



a) Write the equation of motion for small horizontal displacements of the nth mass, ψ_n.

b) Derive a dispersion relation for the propagating modes.

c) What is the range of frequencies (bandwidth) over which waves can propagate along the chain?

The pendulum acts as a spring with constant $\frac{g}{l}$, and the springs provide a force $K(\psi_{n+1} - 2\psi_n + \psi_{n-1})$. This gives an equation of motion

$$m\ddot{\psi}_n + \frac{mg}{l}\psi_n + K(\psi_{n+1} - 2\psi_n + \psi_{n-1}) = 0. \tag{1.5.3}$$

To derive the dispersion relation, assume the disturbances are described by a wave $e^{i(kx-\omega t)}$. We find

$$-m\omega^2 + \frac{mg}{l} + 2K(\cos(ka) - 1) = 0. \tag{1.5.4}$$

Solving gives a dispersion relation

$$\omega = \sqrt{\frac{g}{l} - 4\frac{K}{m}\sin^2\frac{ka}{2}}. \tag{1.5.5}$$

We need real ω in order to get propagating waves. This implies

$$\sin\frac{ka}{2} < \frac{1}{2}\sqrt{\frac{mg}{Kl}}. \tag{1.5.6}$$

So long as the right hand side is less than one, it would seem that there are infinite patches of k which satisfy this. However, the periodicity of the system implies that $k \sim k + \frac{2\pi}{a}$, so this is not the case.

Problem 1.23 (M99M3)
A drum can be considered to be a uniform membrane of mass per unit area ρ, stretched on a rim of radius R. The tension τ, per unit length of the membrane, is so large that it can be considered to be constant even when the membrane is slightly deformed from its equilibrium (flat) shape. (If the membrane is imagined to be cut, then τ is the force which, applied to unit length on either side of the cut will hold the membrane in its place.)

a) How far does the center of the membrane sag below the level of the rim when the drum is held horizontal and the acceleration of gravity is g?

b) Find the lowest vibrational frequency f of the membrane. Ignore gravity in this part of the problem. A reasonable approximate solution will be accepted. However, you may want to know that the first zero of the Bessel function $J_0(x)$, which solves the differential equation $J_0'' + \frac{1}{x}J_0' + J_0 = 0$, is $J_0(0.766\pi) = 0$.

It helps to imagine the drum as a grid of small springs. It is then relatively easy to see that the net force on an infinitesimal mass $\rho\, dA$ is $\tau\nabla^2 z$, where z is the height function on the membrane.

To solve part a), we look for a solution to $\tau\nabla^2 z = \rho g$ with boundary condition $z = 0$ on the circle of radius R. The Laplacian in polar coordinates is

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta}. \tag{1.5.7}$$

Clearly the depression will be spherically symmetric, so we have

$$\frac{\partial^2 z}{\partial^2 r} + \frac{1}{r}\frac{\partial^2 z}{\partial^2 r} = \frac{\rho g}{\tau}. \tag{1.5.8}$$

Clearly we should take $z = ar^2 + b$. Indeed, we can satisfy the equation and the boundary conditions with

$$z(r,\theta) = \frac{\rho g}{4\tau}(r^2 - R^2). \tag{1.5.9}$$

Thus, the membrane sags by $\frac{\rho g R^2}{4\tau}$.

To determine the vibrational frequencies of the membrane, we use Newton's second law on an infinitesimal patch of the drum,

$$\rho\ddot{z} - \tau\nabla^2 z = 0. \tag{1.5.10}$$

We separate z as

$$z(r,\theta,t) = R(r)\Theta(\theta)T(t), \tag{1.5.11}$$

and obtain

$$\frac{T''}{T} - \tau\frac{R'' + \frac{1}{r}R'}{R} - \tau\frac{\Theta''}{r^2\Theta} = 0. \tag{1.5.12}$$

Since the first term is the only time-dependent term, it must in fact be constant, so $T(t) = e^{i\omega t}$ and we have

$$-\tau r^2\frac{R'' + \frac{1}{r}R'}{R} - \tau\frac{\Theta''}{\Theta} = \rho\omega^2 r^2. \tag{1.5.13}$$

Now the second term on the right is the only θ dependent term, so it also must be constant, meaning $\Theta(t) = e^{imt}$ with $m \in \mathbb{Z}$ for single-valuedness. At last we have the radial equation

$$r^2 R'' + rR' + \left(k^2 r^2 - m^2\right) R = 0. \tag{1.5.14}$$

where $k^2 = \frac{\rho}{\tau}\omega^2$. This is the Bessel equation in the dimensionless variable kr, and we will need to satisfy the boundary condition at kR. The first zeroes of the Bessel functions $J_m(x)$ are increasing in m, so we obtain the lowest frequency ω by choosing $m = 0$, such that $R(r) = J_0(kr)$. We then have $k = \frac{0.766\pi}{R}$, so the lowest frequency is

$$\omega = 0.766\pi\sqrt{\frac{\tau}{\rho R^2}}. \tag{1.5.15}$$

So much for springs and bendy solids. Now we'll move on to fluids. Very general fluids are described by the Navier-Stokes equations, which are famously difficult. Usually we can get away with using something like the Euler equations, which describe incompressible and inviscid (zero viscosity) fluids. There is the conservation of mass, which takes the form of the continuity equation

$$\frac{\partial \rho}{\partial t} + \boldsymbol{u} \cdot \nabla \rho = 0, \tag{1.5.16}$$

with ρ the density and $\boldsymbol{u}$ the velocity field. Incompressibility implies that the fluid isn't bunching up anywhere, so

$$\nabla \cdot \boldsymbol{u} = 0. \tag{1.5.17}$$

And finally, Newton's second law becomes a differential equation for the velocity field,

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\frac{\nabla p}{\rho} + \boldsymbol{g}, \tag{1.5.18}$$

where p is the pressure and $\boldsymbol{g}$ is an acceleration coming from a force $\boldsymbol{f} = \rho \boldsymbol{g}$. This alone is sometimes called Euler's equation.

Problem 1.24 (J07M3)
An explosion at time $t = 0$ in an ideal (zero viscosity) incompressible fluid produces a perfectly spherically symmetric expanding bubble of vacuum with radius $R(t)$ (neglect the effect of any gas or vapor inside the bubble). The bubble expands to maximum radius $R_{\max}$ and then collapses. The pressure in the fluid far from the bubble is P_∞, and the mass density of the fluid is ρ. Neglect any effects of surface tension or gravity; assume the bubble remains spherically symmetric at all times, and that the velocity field in the fluid is purely radial.

a) Obtain an expression for the velocity field inside the fluid, and hence get an expression for the total energy (kinetic + potential) of the fluid in terms of R and dR/dt.

b) Obtain an equation of motion for the bubble's radius $R(t)$ of the form

$$\frac{dR}{dt} = f(R). \tag{1.5.19}$$

What is the function f(R)?

c) How long does it take for the bubble to collapse after it reaches its maximum radius? Your answer can contain a finite dimensionless integral whose value you have not obtained.

d) What is the asymptotic behavior of $R(t)$ in the final moments of the bubble's collapse when $R \ll R_{\max}$?

We know the velocity field is radial, with magnitude is $\frac{dR}{dt}$ at the boundary and zero at infinity. The incompressibility equation is

$$\nabla \cdot \boldsymbol{u} = \frac{1}{r^2}\frac{\partial^2}{\partial^2 r}(r^2 u_r) = 0. \tag{1.5.20}$$

This implies $\boldsymbol{u} = \frac{dR}{dt}\frac{R^2}{r^2}\hat{\boldsymbol{r}}$. The kinetic energy is

$$\frac{\rho}{2}\int_R^\infty \left(\frac{dR}{dt}\frac{R^2}{r^2}\right)^2 (4\pi r^2)\, dr = 2\rho\pi R^3 \left(\frac{dR}{dt}\right)^2. \tag{1.5.21}$$

The potential energy comes from the pressure in the fluid. We don't need to worry about pressure at finite distance, because it contributes equally to the potential energy before and after the explosion. All that matters is the net volume $\frac{4\pi}{3}R^3$ which is displaced to infinity, giving

$$U = \frac{4\pi}{3}R^3 P_\infty. \tag{1.5.22}$$

In total we have

$$E = \frac{4\pi}{3}R^3\left(P_\infty + \frac{3}{2}\rho\left(\frac{dR}{dt}\right)^2\right). \tag{1.5.23}$$

To fix the value of the energy, we look at its value at $R = R_{\max}$, $\frac{4\pi}{3}R_{\max}^3 P_\infty$. It follows that

$$\frac{dR}{dt} = \sqrt{\frac{2}{3}\frac{P_\infty}{\rho}\left(\frac{R_{\max}^3}{R^3} - 1\right)}. \tag{1.5.24}$$

The time it takes to collapse is

$$T = \int_0^{R_{\max}} \frac{dR}{dR/dt} = \sqrt{\frac{3\rho}{2P_\infty}}R_{\max}\int_0^1 \frac{u^{3/2}du}{\sqrt{1-u^3}}. \tag{1.5.25}$$

During the death throes of the bubble, when $R \ll R_{\max}$, we have $\frac{dR}{dt} \sim -R^{-3/2}$, which implies $R \sim (T-t)^{2/5}$.

Problem 1.25 (J08M3)
When we derive Newton's equations of motion from a Lagrangian or Hamiltonian, the equations are invariant under time reversal, so that if $x(t)$ is a solution, so is $x(-t)$. If we add terms

corresponding to damping or viscosity, the invariance is broken, and motions become obviously irreversible. Strangely, a form of reversibility is restored for fluid motion in the limit that viscosities are very large.

Consider a fluid with viscosity η and density ρ, and assume that it is incompressible. The equations of motion are the Navier-Stokes equations,

$$\rho\left[\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\right] = \nabla p + \eta\nabla^2\boldsymbol{v}, \tag{1.5.26}$$

$$\nabla\cdot\boldsymbol{v} = 0, \tag{1.5.27}$$

where $\boldsymbol{v}(\boldsymbol{x},t)$ is the velocity of the fluid element at position $\boldsymbol{x}$ at time t, and $p(\boldsymbol{x},t)$ is the pressure. To be concrete, imagine that we have a layer of fluid between two (large) parallel plates, a distance d apart.

a) Let one of the plates move at velocity v_0, with the other plate held fixed. Now the natural unit of length is d, the natural unit of velocity is v_0, and the natural unit of pressure is ρv_0^2. Show that, in these natural units, a single term in the Navier-Stokes equations becomes dominant at large viscosity. Since viscosity has units, "large" means large relative to some characteristic scale η_c, which you should determine.

b) In this limit of large viscosity (usually called the "low Reynolds number" limit, $\mathrm{Re}\equiv\eta_c/\eta$), show that if the plate moves for a time T with velocity v_0, and then with velocity $-v_0$ for an equal time T, all elements of the fluid will be returned exactly to their initial locations, so that motion is reversible. You should show this explicitly for the problem of fluid between two plates (by solving the equations), and give a more general argument (which doesn't require solving the equations).

We start by de-dimensionalizing the Navier-Stokes equations. The unit is pressure per length, so we should divide by $\frac{\rho v_0^2}{d}$ and define $\boldsymbol{V} = \frac{\boldsymbol{v}}{v_0}$, $T = \frac{tv_0}{d}$, $\overline{\nabla} = d\nabla$, $P = \frac{p}{\rho v_0^2}$, and $\overline{\eta} = \frac{\eta}{\eta_c}$ where $\eta_c = \rho v_0 d$. We then have

$$\frac{\partial \boldsymbol{V}}{\partial T} + (\boldsymbol{V}\cdot\overline{\nabla})\boldsymbol{V} = \overline{\nabla}P + \overline{\eta}\overline{\nabla}^2\boldsymbol{V}. \tag{1.5.28}$$

Naturally, for $\overline{\eta}\gg 1$, the viscosity term dominates.

In this limit, the velocity field should solve the Laplace equation at any given time. It is well-known that the Laplace equation with Dirichlet boundary conditions has a unique solution. Furthermore, the equation is linear, and so the velocity field in the bulk will depend linearly on the velocity at the boundary. From this it follows that if we move the boundary in a closed path, every element of fluid will likewise move in a closed path.

To see this explicitly in the case of the plates, let the top plate move in the x direction and let the plates be separated in the y direction. We solve to find

$$V_x = Y^2, \tag{1.5.29}$$

for $Y = y/d$. Restoring dimensions, a fluid element at height y will move $v_0\frac{y^2}{d^2}T$ to the right (no relation to de-dimensionalized T), and then move the same distance to the left.

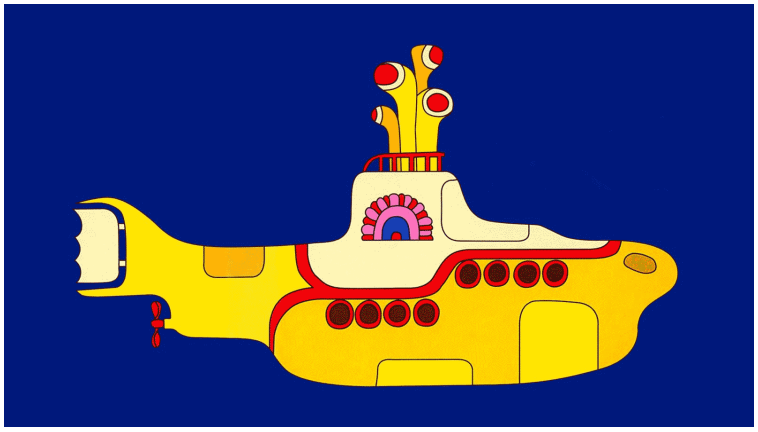


Figure 1.8: Sky of blue and sea of green, in our yellow submarine.

1.6 Non-Inertial Frames

As the great physicist Ringo Starr once said, we all live on a yellow submarine. Some people think this is a statement about the peculiarities of human circumstance, and the need for companionship in this strange experience of life. Fools! "Yellow Submarine" is clearly a song about how the laws of physics should hold in any frame of reference, so long as we make the appropriate changes to dynamic quantities to account for the change of frame.

Newton's laws hold for any "inertial frame," a concept which is ill-defined and doesn't really exist, but you know an inertial frame when you see one. For non-inertial frames, we need to add additional forces to explain away accelerations we observe due to our own accelerating point of view. Some people call these "fictitious forces," but this is an unwarranted attack on Ringo Starr (and also on Einstein and the principle of general relativity).

The simplest case is a uniformly accelerating frame. If our frame has acceleration $\boldsymbol{a}$, then any acceleration $\boldsymbol{a}'$ in an inertial frame is recorded as $\boldsymbol{a}' - \boldsymbol{a}$ in the accelerated frame, so we need to add a force $-m\boldsymbol{a}$ to account for the change.

A more interesting case is when the frame is rotating. The principle is the same, but now we need to account for changes in our basis vectors. From (1.4.33), we have the velocity transformation

$$\boldsymbol{v} = \boldsymbol{v}_{\rm rot} + \boldsymbol{\omega} \times \boldsymbol{r}. \tag{1.6.1}$$

Taking another time derivative and using (1.4.33) gives

$$\boldsymbol{a}_{\rm rot} = \boldsymbol{a} - 2\boldsymbol{\omega} \times \boldsymbol{v} - \boldsymbol{\omega} \cdot (\boldsymbol{\omega} \times \boldsymbol{r}). \tag{1.6.2}$$

The second term on the right is the Coriolis force, and the third term is centrifugal acceleration. In situations where ω is small compared to relevant frequency scales, such as for the rotation of the Earth, the Coriolis term is dominant over the centripetal term.

Problem 1.26 (M02M3)
A particle is dropped vertically in the Earth's gravitational field at latitude λ. Assume it feels an air drag $F = kv^2$. Due to the Coriolis effect, it will undergo a horizontal deflection.

a) Initially neglect the Earth's rotation. Find an explicit equation for the vertical velocity.

b) Working at leading order in the Earth's angular velocity ω, and using the result you just derived, find the horizontal velocity as a function of time.

c) What is the velocity at $t \gg \sqrt{\frac{m}{gk}}$?

If we ignore the Earth's rotation, we have

$$m\dot{v}_z = mg - kv_z^2, \tag{1.6.3}$$

implying

$$v_z(t) = \sqrt{\frac{mg}{k}} \tanh\left(\sqrt{\frac{kg}{m}}t\right). \tag{1.6.4}$$

The magnitude of the Coriolis acceleration will be $2\boldsymbol{\omega} \times \boldsymbol{v} = 2\omega v_z(t) \cos\lambda$, so

$$\dot{v}_h(t) = 2\omega\cos\lambda\sqrt{\frac{mg}{k}} \tanh\left(\sqrt{\frac{kg}{m}}t\right). \tag{1.6.5}$$

Carrying out the integral, we find

$$v_h(t) = 2\frac{m\omega}{k}\cos\lambda \ln\cosh\left(\sqrt{\frac{kg}{m}}t\right). \tag{1.6.6}$$

At very large times, the vertical velocity is asymptotic to $\sqrt{\frac{mg}{k}}$, whereas the horizontal velocity grows linearly with t as $2\omega t\sqrt{\frac{mg}{k}}$. Thus, if a particle falls for a very long time – several days, so that $\omega t \gg 1$ – it will begin moving almost entirely horizontally. The total speed will approach $\sqrt{\frac{mg}{k}(4\omega^2 t^2 + 1)}$.

Problem 1.27 (J12M3)
An upright cylindrical bucket has radius R and its rim is at height H. This bucket is placed on a horizontal surface and filled to a height $h < H$ with incompressible water. The bucket is then rotated at an angular frequency ω about a vertical axis that goes through the center of the bucket. Let g be the acceleration due to gravity, and assume that the water is simply rotating with the same angular velocity as the bucket.

a) Assuming that ω is small enough that the water does not reach the rim of the bucket or reveal the bottom of the bucket, find an expression for the height of the water's surface as a function of the radial distance from the central axis of rotation.

b) Quantitatively, what are the conditions on ω such that water neither spills over the rim of the bucket nor reveals the bottom of the bucket?

We are assuming the water rotates with the bucket, so $\boldsymbol{v} = 0$ in the rotating frame and there is no Coriolis force. We only need to worry about the centrifugal force, which will be directed outwards with magnitude $\omega^2 r$, where r is the distance from the rotation axis. We can represent this with a potential $-\frac{1}{2}\rho\omega^2 r^2$. The surface of the water will be at an equipotential surface with respect to the total potential $\rho g h - \frac{1}{2}\omega^2 r^2$, which gives

$$z(r) = \frac{\omega^2}{2g} r^2 + z_0. \tag{1.6.7}$$

To determine the constant z_0, we use the conservation of the water volume, assuming $z_0 \geq 0$. We have

$$\pi R^2 h = \int_0^R z(r)(2\pi r)\, dr = \frac{\pi\omega^2 R^4}{4g} + \pi R^2 z_0. \tag{1.6.8}$$

Solving, we find $z_0 = h - \frac{\omega^2 R^2}{4g}$, which means the greatest height is $z(R) = \frac{\omega^2 R^2}{4g}$. In order to keep the the bottom of the bucket covered and to keep the water in the bucket, we need

$$\omega < \sqrt{\frac{4g}{R^2} \min(h, H - h)}. \tag{1.6.9}$$

The hydrostatic equilibrium condition used in the previous problem, in which the surface of a fluid traces an equipotential surface, can be useful in more complicated scenarios.

Problem 1.28 (J01M2)
The following two problems relate to a calculation of the angular frequency Ω of free precession of a planet or star whose angular frequency of rotation about its axis is ω. The problems themselves are independent.

Suppose that the density ρ of the object is uniform, and that its shape can be determined by the condition of hydrostatic equilibrium. Deduce an expression for the (small) quantity $\epsilon(\omega, M, r_p)$ that relates the equatorial radius r_E to the polar radius r_P by the form $r_E = r_P(1 + \epsilon)$, where $M \approx 4\pi\rho(r_P^3)/3$ is the mass of the object.

The rotation of the Earth about its axis generates a centrifugal potential $-\frac{1}{2}\rho\omega^2 s^2$, where s is the distance from the rotation axis. There is also the potential due to gravity, which we approximate by $-\frac{GM}{r}$. The equatorial bulge will be such that the increase in gravitational potential balances the negative centrifugal potential. Since $s = r$ at the equator, this condition gives

$$\frac{GM}{r_P} - \frac{GM}{r_E} = \frac{1}{2}\omega^2 r_E^2. \tag{1.6.10}$$

Assuming $r_E = r_P(1+\epsilon)$ with small ϵ, this becomes

$$\frac{GM}{r_P}\epsilon = \frac{1}{2}\omega^2 r_P^2(1+\epsilon)^2, \tag{1.6.11}$$

so

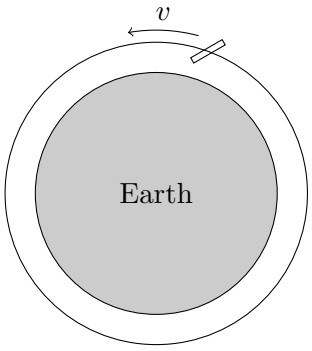
$$\epsilon = \frac{g-\omega^2 r_P}{\omega^2 r_P} - \frac{\sqrt{(g(g-2r_P\omega^2)}}{r_P\omega^2}, \tag{1.6.12}$$

where $g = \frac{GM}{r_P^2}$. We have chosen this root because it is very small for $g \gg \omega^2 r_P$, which is the case in practice. Indeed, define $\alpha = \frac{g-\omega^2 r_P}{\omega^2 r_P}$. Then we have

$$\epsilon = \alpha - \sqrt{(\alpha+1/2)(\alpha-1/2)} \approx \frac{1}{8\alpha} = \frac{\omega^2 r_P}{2(g-\omega^2 r_P)} \approx \frac{\omega^2 r_P^3}{2GM}. \tag{1.6.13}$$

Rotating frames are especially well-suited for problems involving orbits because everything is going in a circle anyway. The competition between centrifugal forces and gravitational forces is behind the effective potential given in Sec. 1.3, and we can address it more explicitly by moving to a frame corotating with an orbiting body.

Problem 1.29 (M05M3)
The orientation of a satellite in low-Earth orbit can be stabilized with gravity gradients. Let the satellite be a long cylinder of length L and radius a ($a \ll L$). The distribution of the mass m is uniform within the cylinder. The satellite is in a circular orbit around Earth with a period $T_{\rm or}$. Express your answers to parts b), c) and d) in terms of $T_{\rm or}$, L, and a.



a) What is the stable equilibrium orientation, for which the satellite appears to be at rest in a coordinate system rotating with the same angular velocity as the orbital motion?

b) What is the period T_π of small oscillations about equilibrium in the orbital plane?

c) What is the period T_σ of small oscillations about equilibrium perpendicular to the orbital plane?

d) The satellite is given a small angular speed ω around its long axis. To first order in ω, find the new stable equilibrium orientation of the satellite, where the direction of the long axis appears to be fixed in the rotating coordinate system.

In a frame rotating with the satellite, it has no velocity, so there is no Coriolis force. We compute the energy of the satellite when it is tilted at angle θ with respect to a tangent to its orbit. Let r be the distance between the center of masses of the satellite and the Earth. The potential energy due to gravity is

$$U_g = -\int_{-L/2}^{L/2} \frac{GM(m/L)}{\sqrt{r^2+x^2+2rx\sin\theta}}\,dx \approx -\frac{GMm}{r}\left(1+\frac{L^2}{24r^2}(3\sin^2\theta-1)\right), \tag{1.6.14}$$

and the part due to the centrifugal potential is

$$U_c = \int_{-L/2}^{L/2} \frac{(m/L)\omega^2(a^2+x^2+2ax\sin\theta)}{2}\,dx = \frac{1}{2}m\omega^2 r^2\left(1+\frac{L^2}{24r^2}\right). \tag{1.6.15}$$

There is no θ-dependence, so the centrifugal potential is unimportant in this case. The angular velocity of the orbit satisfies $\omega^2 = \frac{GM}{a^3}$, so the θ-dependent part of the potential is

$$U(\theta) = -\frac{GMmL^2}{8r^3}\sin^2\theta. \tag{1.6.16}$$

This implies that the satellite will be in equilibrium at $\theta = \pi$, oriented vertically.

The moment of inertia of the satellite is $\frac{1}{3}mL^2$. Thus, the period for small oscillations in the orbital plane is

$$T_\pi = 2\pi\sqrt{\frac{12r^3}{GM}} = \frac{2}{\sqrt{3}}T_{\text{or}}. \tag{1.6.17}$$

For oscillations perpendicular to the orbital plane, the form of the centrifugal potential will change slightly. We have

$$U_c'(\phi) = \int_{-L/2}^{L/2} \frac{(m/L)\omega^2(r+x\sin\phi)^2}{2}\,dx = \frac{1}{2}m\omega^2 r^2\left(1+\frac{L^2}{12r^2}\sin^2\phi\right). \tag{1.6.18}$$

The total potential for an angle ϕ perpendicular to the orbital plane is then

$$U'(\phi) = -\frac{GMmL^2}{12r^3}\sin^2\phi, \tag{1.6.19}$$

giving a period

$$T_\sigma = \sqrt{2}T_{\text{or}}. \tag{1.6.20}$$

To treat the spinning satellite, we compute the kinetic energy. The angular velocity will be

$$(\Omega\sin\phi+\omega)\boldsymbol{e}_1 + \Omega\cos\phi\boldsymbol{e}_2, \tag{1.6.21}$$

where $\Omega = \sqrt{\frac{GM}{r^3}}$ is the orbital angular velocity, $\boldsymbol{e}_1$ is directed along the satellite, and $\boldsymbol{e}_2$ is perpendicular to it. The corresponding moments of inertia along these principal axes are $\frac{1}{2}ma^2$ and $\frac{1}{3}mL^2$, so the kinetic energy of rotation is

$$T = \frac{1}{4}ma^2(\Omega\sin\phi+\omega)^2 + \frac{1}{3}mL^2\Omega^2\cos^2\phi. \tag{1.6.22}$$

The θ-dependent pieces of the energy are

$$U = \frac{ma^2}{4}(\Omega^2 \sin^2\phi + 2\omega\Omega\sin\phi) + \frac{mL^2}{3}\Omega^2\cos^2\phi - \frac{m\Omega^2L^2}{12}\sin^2\phi. \tag{1.6.23}$$

Assuming small ϕ, this reduces to

$$U = \frac{m\Omega^2}{4}\left(a^2 - \frac{L^2}{3}\right)\phi^2 + \frac{ma^2}{2}\omega\Omega\phi. \tag{1.6.24}$$

This is minimized by the equilibrium angle

$$\phi = \frac{\omega}{\Omega}\left(\frac{L^2}{3a^2} - 1\right)^{-1}. \tag{1.6.25}$$

Problem 1.30 (M03M1)
The Earth is in a circular orbit of angular frequency ω about the Sun. The Sun is so much more massive that the Earth that, for our purposes, it may take to sit at rest at the center of our coordinate system. Lagrange discovered that there exist a certain number of equilibrium points at which an artificial satellite of negligible mass can orbit the Sun with the same frequency ω as the Earth (while maintaining a fixed distance from the Earth and the Sun). Such orbits are ideally suited for space-based observatories of various kinds. We will explore some of the properties of the 'Lagrange points' in this problem.

a) Consider points on the line that runs from the Sun through the Earth. This line is of course stationary in the reference frame that rotates with the Earth in its orbit. Show that there is one point on this line outside the Earth's orbit where a test particle may sit at equilibrium in the rotating frame. This point is commonly designated as the L2 Lagrange point. (There is a similar Lagrange point, L1, inside the Earth's orbit as well.)

b) Give an approximate expression, correct to leading order in the small quantity $\beta = M_e/M_s$, for the distance from the Earth to the L2 Lagrange point described above. Express your answer in terms of the masses and R, the Earth-Sun distance. The Wilkinson Microwave Anisotropy Probe (WMAP) is stationed at L2: using $\beta \approx 3 \times 10^{-6}$ and $R = 1.5 \times 10^8\,\mathrm{km}$, find the distance from Earth to WMAP.

c) Determine whether the L2 equilibrium point is stable or unstable against small perturbations in position along the Earth-Sun line.

We use a frame which co-rotates with the Earth around the Sun. The potential for $r > R$ can then be given in three pieces:

$$U(r) = -\frac{GM_sm}{r} - \frac{GM_em}{r-R} - \frac{1}{2}m\omega^2r^2. \tag{1.6.26}$$

Setting the derivative to zero, we will find equilibria wherever

$$\frac{GM_s}{r^2} + \frac{GM_e}{(r-R)^2} - \omega^2 r = 0. \tag{1.6.27}$$

At $r = R + \epsilon$, the first two terms are very large and decreasing, while the third term is negative and decreasing. Thus, as we increase r to infinity, we will cross only one zero, which is the L2 point.

To find the distance to this point, $u = r - R$, we need to solve

$$\frac{1}{(R+u)^2} + \frac{\beta}{u^2} - \frac{R+u}{R^3} = 0, \tag{1.6.28}$$

where we have used $\omega^2 = \frac{GM_s}{R^3}$ to eliminate ω. Since β is small, we can rearrange this to read

$$\left(1 + \frac{u}{R}\right)^3 = 1 + \beta\frac{R^2}{u^2}\left(1 + \frac{u}{R}\right)^2 \tag{1.6.29}$$

and then take the approximate solution

$$u = \left(\frac{\beta}{3}\right)^{1/3} R + \mathcal{O}\left(\beta^2\right). \tag{1.6.30}$$

Using actual numbers, like godless heathens, we find that WMAP is about 1.5×10^6 km away.

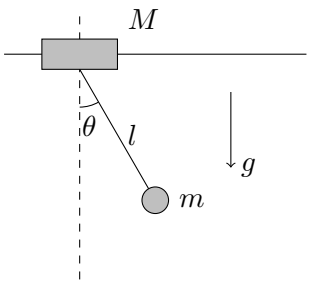
To determine stability, we look at the second derivative of the potential,

$$\frac{GM_s}{R^3}\left(-\frac{2}{(1+u/R)^3} - \frac{2\beta}{(u/R)^3} - 1\right). \tag{1.6.31}$$

Every term is negative, so this Lagrange point is unstable. Apparently satellites at L2 have to undergo constant course corrections to keep themselves from falling off the top of the hill. Sisyphus is real and he's in the sky.

1.7 Additional Problems

Problem 1.31 (J08M1)
A plane pendulum consists of a bob of mass m suspended by a massless rigid rod of length l that is hinged to a sled of mass M. The sled slides without friction on a horizontal rail. Gravity acts with the usual downward acceleration g.



a) Taking x and θ as generalized coordinates, write the Lagrangian for the system.

b) Derive the equations of motion for the system.

c) Find the frequency ω for small oscillations of the bob about the vertical.

d) At time $t = 0$ the bob and the sled, which had previously been at rest, are set in motion by a sharp tap delivered to the bob. The tap imparts a horizontal impulse $\Delta P = F\Delta t$ to the bob. Find expressions for the values of $\dot{x}$ and $\dot{\theta}$ just after the impulse.

The position of the bob is $(x + l\sin\theta, -l\cos\theta)$. Therefore, the Lagrangian is

$$L = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x}^2 + l^2\dot{\theta}^2 + 2l\dot{x}\dot{\theta}\cos\theta) + mgl\cos\theta. \tag{1.7.1}$$

Therefore, the equations of motion are

$$\frac{d}{dt}\Big((M+m)\dot{x} + ml\dot{\theta}\cos\theta\Big) = 0, ml^2\ddot{\theta} + ml\ddot{x}\cos\theta - ml\dot{x}\dot{\theta}\sin\theta + mgl\sin\theta \quad = 0. \tag{1.7.2}$$

In matrix form and assuming small angles,

$$\begin{pmatrix} M+m & ml \\ ml & ml^2 \end{pmatrix}\begin{pmatrix} \ddot{x} \\ \ddot{\theta} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & mgl \end{pmatrix}\begin{pmatrix} x \\ \theta \end{pmatrix}. \tag{1.7.3}$$

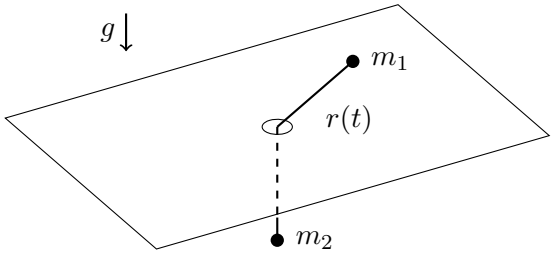
We have

$$\begin{pmatrix} M+m & ml \\ ml^2 & 0 \end{pmatrix}^{-1}\begin{pmatrix} 0 & 0 \\ 0 & mgl \end{pmatrix} = \begin{pmatrix} 0 & -g/l \\ 0 & (1+m/M)g/l \end{pmatrix}, \tag{1.7.4}$$

The frequency is therefore $\omega = \sqrt{\left(1+\frac{m}{M}\right)\frac{g}{l}}$. The zero mode corresponds to uniform acceleration of the sled.

The horizontal force acting on the sled due to the bob is proportional to $\sin\theta$. Thus, during the tap the force vanishes, and thus the sled is not accelerated. All the momentum goes into the bob, so we have $\dot{\theta} = \frac{F\Delta t}{m\ell}$ immediately after the tap.

Problem 1.32 (J00M3)
A mass m_1 slides without friction on a horizontal table. The mass is tied to a string with negligible mass that passes without friction through a small hole. A mass m_2 is tied to the other end of the string. The uniform gravitational acceleration g is normal to the table.



The orbit of m_1 is only slightly perturbed from circular. The masses m_1 and m_2 are chosen so the orbit is closed, with one maximum and one minimum of the distance $r(t)$ of m_1 from the hole, when computed to first order in the departure from a circular orbit. Find m_2 in terms of the other parameters.

The effective potential is given by

$$V_{\text{eff}}(r) = \frac{L^2}{2m_1 r^2} + m_2 g r. \tag{1.7.5}$$

The minimum is at $r_0 = \left(\frac{L^2}{m_1 m_2 g}\right)^{1/3}$. Clearly the $m_2 g r$ potential will not contribute a second derivative, so we have

$$V_{\text{eff}}(r) \approx V_{\text{eff}}(r_0) + \frac{3L^2}{m_1 r_0^4}(r - r_0)^2. \tag{1.7.6}$$

The resulting restoring force acts on the combined mass $m_1 + m_2$. Thus, the frequency of oscillations about this equilibrium is

$$\omega = \sqrt{3}\frac{L}{\sqrt{m_1(m_1 + m_2)}r_0^2}. \tag{1.7.7}$$

In order to have an orbit of the form described, we need $\omega = \frac{L}{m_1 r_0^2}$. This implies $m_2 = 2m_1$.

Problem 1.33 (J03M1)
This problem is about scattering by an attractive potential.

a) Consider a particle with energy E and $z < 0$ approaching the $z = 0$ plane at an angle θ_1 to the z-axis. Find the angle θ_2 that it makes to the z axis after passing through the $z = 0$ plane if $V = 0$ for $z < 0$ and $V = -V_0$ (constant) for $z > 0$.

b) Apply your result to a uniform beam of particles scattered by the attractive potential

$$V(r) = -V_0 \quad r < a, \qquad V(r) = 0 \quad r > a. \tag{1.7.8}$$

Determine the differential cross section. (Recall that the definition of the differential cross section is $\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\frac{db}{d\theta}$, where b is the impact parameter and θ the scattering angle.)

This form of scattering is effectively the same as refraction. Momentum in the plane is conserved, and we know the total speed before and after the scattering, so

$$\sqrt{\frac{2(E+V_0)}{m}}\sin\theta_2 = \sqrt{\frac{2E}{m}}\sin\theta_1 \implies \sin\theta_2 = \sqrt{\frac{1}{1+V_0/E}}\sin\theta_1. \tag{1.7.9}$$

When a particle at impact parameter b strikes the sphere, it will have $\sin\theta_1 = \frac{b}{R}$, so

$$\theta_2 = \sin^{-1}\left(\frac{b}{R}\sqrt{\frac{1}{1+V_0/E}}\right). \tag{1.7.10}$$

The deflection of the particle is $\theta_1 - \theta_2$. If we continue its trajectory through the sphere, we find that it must strike the interior wall at the same angle θ_2, which means it will exit at angle θ_1. Thus, the total deflection of the particle due to the potential is $\theta = 2(\theta_1 - \theta_2)$, or

$$\theta = 2\left(\sin^{-1}\frac{b}{R} - \sin^{-1}\left(\frac{b}{R}\sqrt{\frac{1}{1+V_0/E}}\right)\right). \tag{1.7.11}$$

Now we solve for b and differentiate with respect to θ. God have mercy on my soul. Actually, it's not so bad – rearranging and taking a sin, we get

$$\sqrt{1-\frac{b^2}{R^2}}\sin\frac{\theta}{2} - \frac{b}{R}\cos\frac{\theta}{2} = -n\frac{b}{R}, \tag{1.7.12}$$

where we have defined $n = \sqrt{\frac{1}{1+V_0/E}}$. Playing with this a bit more, we get

$$b = \frac{R\sin\frac{\theta}{2}}{\sqrt{n^2 - 2n\cos\frac{\theta}{2} + 1}}, \tag{1.7.13}$$

and so

$$\frac{db}{d\theta} = \frac{1}{2}\frac{\left(n^2+1\right)R\cos\frac{\theta}{2} - nR\left(1+\cos^2\frac{\theta}{2}\right)}{\left(n^2 - 2n\cos\frac{\theta}{2}+1\right)^{3/2}}. \tag{1.7.14}$$

This gives a differential cross section of

$$\frac{d\sigma}{d\Omega} = \frac{R^2}{4\cos\frac{\theta}{2}}\frac{\left(n^2+1\right)\cos\frac{\theta}{2} - n\left(1+\cos^2\frac{\theta}{2}\right)}{\left(n^2 - 2n\cos\frac{\theta}{2}+1\right)^{2}}. \tag{1.7.15}$$

Chapter 2

Electromagnetism

Look at the person to your left. Now look at the person to your right. According to the latest statistics, there's a 100% chance you just observed electromagnetic fields, twice.

In this chapter we'll be concerned with James Clerk Maxwell's equations, using terminology and formalism largely invented and promoted by Oliver Heaviside, involving contributions from a number of other scientists, many of whom had beards somewhere on the wide-ranging Heaviside-Maxwell spectrum. One of particular interest is Hendrik Lorentz; even modern experiments cannot rule out the hypothesis that Lorentz was Santa Claus, and used his namesake contraction effect to squeeze down chimneys and deliver presents. I mean, look at him.



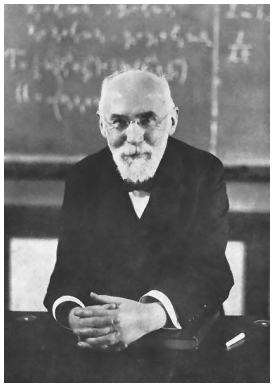


Figure 2.1: (Left) Oliver Heaviside, who formulated electrodynamics in terms of fields and fluxes, and who owned some kind of beard trimming device; (Center) James Clerk Maxwell, who was the first to write down equations equivalent to the modern theory of electrodynamics, and who owned no such beard trimming device; (Right) Hendrik Lorentz, a.k.a. Santa Claus, who helped to understand the relativistic nature of electrodynamics and who gives his name to the Lorentz force law.

In 2.1, we'll look at electric fields on their own, and in 2.2, we'll do another one-on-one session with magnetic fields. In 2.3 we consider what happens when these ghosty fields get involved with actual stuff, and in 2.4 we'll allow the fields to get involved with each other. If the relationship works out, the fields form light, so we'll discuss radiation in 2.6.

2.1 Electrostatics

Stuff in the world can be charged, and opposite charges attract while like charges repel. Much like gravity, this attraction follows an inverse square law, in this case called Coulomb's law:

$$\boldsymbol{F} = \frac{q_1 q_2}{r^2}\hat{\boldsymbol{r}}. \tag{2.1.1}$$

This force comes from the potential energy

$$U(r) = \frac{q_1 q_2}{r}. \tag{2.1.2}$$

Note that the potential is positive for like charges and negative for opposite charges.

With gravity, we found it useful to normalize the potential energy between two masses by the mass of one of them, in order to end up with a gravitational potential that was intrinsic to the other mass. We can do the same here, normalizing by charge. The electric potential due to a charge q is given by

$$\phi(r) = \frac{q}{r}. \tag{2.1.3}$$

Already at this point, it's time for a heart-to-heart about units. Some readers may have expected to see some factors of $4\pi\epsilon_0$ in these equations. Such factors will not be appearing. We are working in Gaussian units, the morally superior alternative to SI units. In this system, charge has units of $(\text{length})^{3/2}(\text{mass})^{1/2}(\text{time})^{-1}$, so that (2.1.1) is dimensionally correct all on its own.

Note that this is distinct from so-called natural units, where certain legitimate physical parameters are taken to be 1 for the sake of expediency, and restored based on dimensional considerations later, so that in the meantime equations appear to have dimensional issues. This is not a mere calculation technique, and we are not saving our ϵ_0's for later. They will never appear, because they are stupid and unnecessary. Same with the μ_0's. Indeed, that's the stupidest part, having different constants for electricity and magnetism – that's like measuring north-south distances in furlongs and east-west distances in kilometers. There is no situation, not for theory or for experiment, where that kind of shenanigan makes any sense whatsoever.

Also, for what it's worth, the SI system *is* a historical accident but not in the way you might think. The SI system was adopted in the early 20^{th} century, whereas Gauss had this system worked out in the mid-19^{th}. The problem was that someone else came along and made some competing system, and then there was confusion between the two systems, and so someone proposed *dreaming up a whole new unit* to somehow resolve the confusion. And apparently that argument won the day.

Saltiness aside, let's continue. From the electric potential, which is a potential energy per unit test charge, we can take a negative gradient to derive the electric field,

$$\boldsymbol{E} = -\nabla\phi, \tag{2.1.4}$$

which is a force per unit charge. We could have defined an analogous field for gravity, but we didn't. That's because the electric field $\boldsymbol{E}$ will actually obey interesting dynamics, whereas in gravity we have to work much (much) harder to work out the dynamical degrees of freedom.

We defined the potential for a point particle, but the real world is better described by charge distributions ρ. The potential in (2.1.3) satisfies $\nabla^2\phi = -4\pi q\delta(x)$, so by analogy we have

$$\nabla \cdot \boldsymbol{E} = 4\pi\rho. \tag{2.1.5}$$

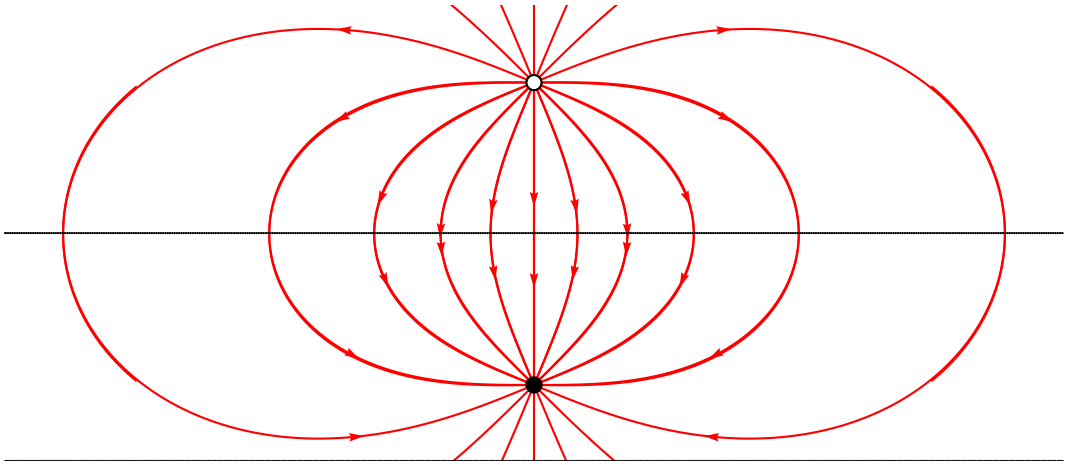


Figure 2.2: If a negative image charge is placed opposite the real positive charge, the resulting electric field is perpendicular to the conductor.

This is known as Gauss's law.

Gauss's law is known to mathematicians as a Poisson equation. Since $\frac{1}{4\pi r}$ is a Green's function for the Laplace operator, the solution is

$$\phi(\boldsymbol{r}) = \int d\boldsymbol{r'} \frac{\rho(\boldsymbol{r'})}{4\pi|\boldsymbol{r} - \boldsymbol{r'}|}. \tag{2.1.6}$$

This is sometimes useful, sometimes not, since we don't always know the charge distribution *a priori*. But it does tell us that the solution is unique (so long as we fix the boundary condition $\phi = 0$ at infinity).

Uniqueness of solutions means in particular that, if we can dream up just one solution which works, then we're done. This can be especially useful in the presence of conductors. We'll deal with matter in detail in Sec. 2.3, but conductors are simple: they let charge move all it wants. If an electric field impinges on a conductor, the charge in the conductor will move in response to it. This continues until the electric field vanishes within the conductor and is perpendicular to its surface, so that the charges in the conductor are no longer pushed around. Put another way, in equilibrium the potential should be constant throughout the conductor.

If we have a point charge sitting next to a conductor, then the charge will move all around in the conductor until the electric field is perpendicular to its surface. We don't know what that charge configuration will look like, so (2.1.6) can't help us. But if we can make a really good guess, uniqueness of solutions can help us. For a conductor which forms an infinite plane, and a charge sitting above it, a really good guess would be to forget the conductor, and in its place, imagine an opposite charge sitting on the opposite side of the plane of the conductor. We would then get an electric field above the conductor which is everywhere perpendicular to it (see Figure 2.2. This satisfies the boundary conditions, so by uniqueness, it's correct.

Problem 2.1 (M09E3)
A point charge Q_1 is located a distance d from the center of a thin, conducting spherical shell of radius R ($d > R$).

a) If the conducting sphere were temporarily grounded, what would be the magnitude Q_0 and distance from the origin d_0 of the image charge?

b) Now the ground connection is removed and the conducting sphere is insulated from the ground, leaving the total net charge Q_0 distributed on the surface. If an additional amount of charge $\Delta Q = Q - Q_0$ is placed on the surface of the conducting sphere, how will the excess charge distribute itself on the surface?

c) What must the net charge on this shell Q be so that there is no net force between the point charge Q_1 and the shell?

If the method of images is to work at all, we at least need to be able to set $V = 0$ on the conductor along the line formed by the origin and the charge Q_1. This gives two equations for the two points on the sphere:

$$\frac{Q_1}{d-R} + \frac{Q_0}{R-d_0} = 0, \tag{2.1.7}$$

$$\frac{Q_1}{d+2R} + \frac{Q_0}{R+d_0} = 0. \tag{2.1.8}$$

Solving these equations gives

$$Q_0 = -\frac{R}{d}Q_1, \qquad d_0 = \frac{R^2}{d}. \tag{2.1.9}$$

We now check that such an image charge actually works, in the sense that it gives $V = 0$ on the entire sphere. We can work in spherical coordinates where $\theta = 0$ is the direction of Q_1. Then the potential on the sphere is

$$V(\theta) = \frac{Q_1}{\sqrt{R^2 + d^2 - 2Rd\cos\theta}} + \frac{Q_0}{\sqrt{R^2 + d_0^2 - 2Rd_0\cos\theta}}. \tag{2.1.10}$$

Substituting the image charge data, this quickly simplifies to zero, so indeed we have found an image charge presentation of the solution to the problem.

We can use a similar trick to answer the second part. We know that the charge Q_0 is distributed on the surface in some oh-so-special way as to make the potential uniform on the whole sphere. If we added some extra charge ΔQ uniformly to the surface, then the potential would still be uniform. By uniqueness, the excess charge will indeed be uniformly distributed.

With no excess charge, the point charge would feel the force it would feel due to the image charge,

$$F = -\frac{Q_0 Q_1}{(d-d_0)^2} = -Q_1^2 \frac{Rd}{(d^2-R^2)^2}. \tag{2.1.11}$$

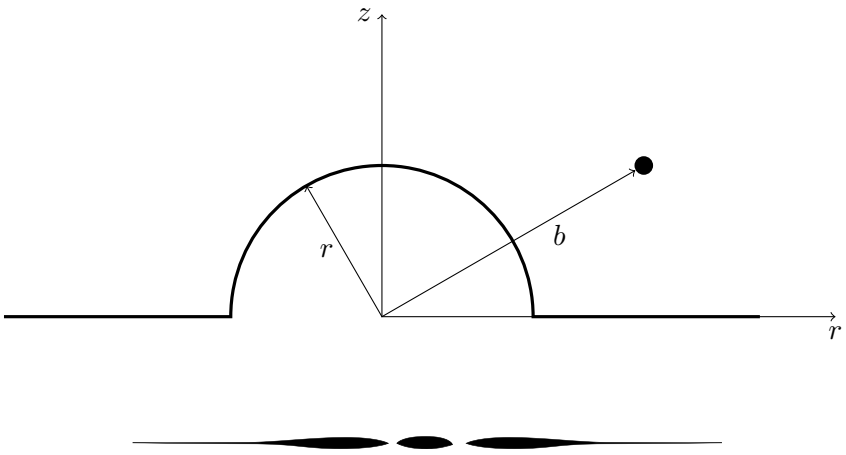
In order to make the net force zero, we should repel F by the same amount. A uniform charge ΔQ on the sphere will have the same electric field as a point charge ΔQ at the origin, so

$$\frac{Q_1 \Delta Q}{d^2} = Q_1^2 \frac{Rd}{(d^2-R^2)^2} \implies \Delta Q = Q_1 \frac{Rd^3}{(d^2-R^2)^2}. \tag{2.1.12}$$

Thus, the net charge is

$$Q = Q_0 + \Delta Q = Q_1 \frac{R}{d}\left(\frac{1}{(1-R^2/d^2)^2} - 1\right). \tag{2.1.13}$$

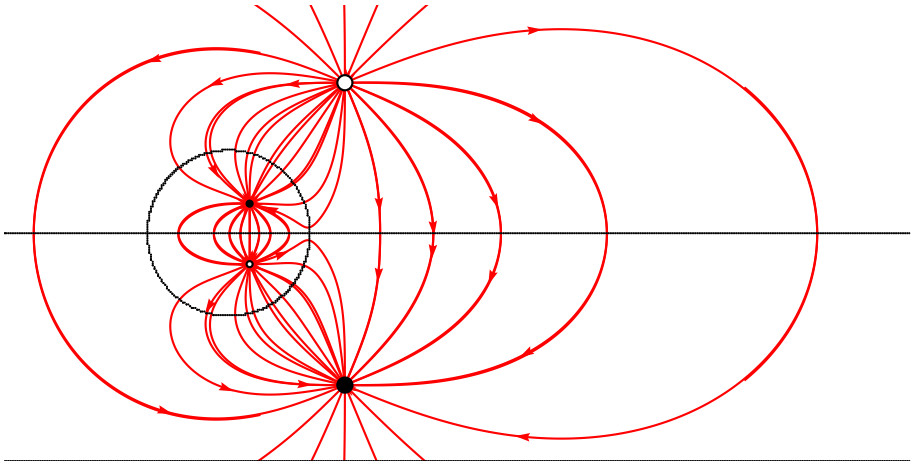
Problem 2.2 (J03E2)
Find the electric potential in cylindrical coordinates $\phi(r, \theta, z)$ when a charge q is located at $(r_0, z_0 > 0)$ and there is a grounded conducting plane at $z = 0$ that has a conducting hemispherical boss of radius $R < b = \sqrt{r_0^2 + z_0^2}$ whose center is at the origin. A side view of the boss and conducting plane is shown in the picture below. What is the electrostatic force on the charge q in part b) for the case that $r_0 = 0$?[1]



We can start by putting an image charge $-\frac{r}{b}q$ at a distance $\frac{r^2}{b}$ from the origin, as in the previous problem. We know this gets us $V = 0$ on the hemispherical boss, but then we have to worry about the plane. Since Gauss's law is linear, we can think of the total field as a superposition of the field due to the original charge and the field due to the image charge. This suggests that we place two more image charges below the plane. In total, we have four charges: q at (r_0, z_0), $-\frac{r}{b}q$ at $\left(\frac{r^2}{b}\frac{r_0}{b}, \frac{r^2}{b}\frac{z_0}{b}\right)$, $-q$ at $(r_0, -z_0)$, and $\frac{r}{b}q$ at $\left(\frac{r^2}{b}\frac{r_0}{b}, -\frac{r^2}{b}\frac{z_0}{b}\right)$. The potential due to the first charge is

$$\phi_1(r, \theta, z) = \frac{q}{\sqrt{r^2 + r_0^2 - 2rr_0\cos\theta + (z - z_0)^2}}, \tag{2.1.14}$$

and the total potential can be found by adding terms of this form.



[1]This problem had an additional part which was nearly identical to the previous problem, and so it is omitted here.

Figure 2.3: The field lines for the charge and image charges in this problem, with the full $\phi = 0$ equipotential drawn.

When $r_0 = 0$, all these charges end up on the z axis, so it's simple to compute the force on the actual charge:

$$\boldsymbol{F} = q\hat{\boldsymbol{z}}\left(-\frac{r}{z_0}q\frac{1}{(z_0 - r^2/z_0)^2} + \frac{r}{z_0}q\frac{1}{(z_0 + r^2/z_0)^2} - q\frac{1}{4z_0^2}\right) \tag{2.1.15}$$

$$= -\frac{q^2}{4z_0^2}\hat{\boldsymbol{z}}\left(\frac{16(r/z_2)^2}{(1 - r^4/z_0^4)^2} + 1\right). \tag{2.1.16}$$

Problem 2.3 (M06E1)
An uncharged metal sphere of radius R is placed inside an otherwise uniform electric field $\boldsymbol{E} = E_0\hat{\boldsymbol{z}}$.

a) Find the electrostatic potential in the region outside the sphere.

b) Find the induced charge density on the surface of the sphere.

The electric field between two opposite charges is close to uniform, as shown in Figure 2.2. Its magnitude, if the charges are at $(0, 0, \pm d)$, is

$$|\boldsymbol{E}| = \frac{2q}{d^2}. \tag{2.1.17}$$

So, we could think of the uniform field $\boldsymbol{E} = E_0\hat{\boldsymbol{z}}$ as the result of a positive charge q at $(0, 0, -d)$ and a charge $-q$ at $(0, 0, d)$, where we take the limit as $d \to \infty$ and keep $\frac{2q}{d^2} = E_0$.

With this approach, we can use image charges to find the potential outside the sphere. Let's pretend for a moment that the sphere is grounded. The image charges will be located at $(0, 0, \pm R^2/d)$ and will have charges with magnitude $q\frac{R}{d}$. The total potential will be

$$\phi(r, \theta, z) = \frac{q}{\sqrt{r^2 + (z + d)^2}} - \frac{q}{\sqrt{r^2 + (z - d)^2}} - \frac{qR/d}{\sqrt{r^2 + (z + R^2/d)^2}} + \frac{qR/d}{\sqrt{r^2 + (z - R^2/d)^2}}. \tag{2.1.18}$$

Expanding this to leading order in $\frac{1}{d}$, we find

$$\phi(r, \theta, z) = -\frac{2q}{d^2}z\left(1 - \frac{R^3}{(r^2 + z^2)^{3/2}}\right). \tag{2.1.19}$$

In the limit $d \to \infty$, this becomes

$$\phi(r, \theta, \varphi) = -E_0 r\cos\theta\left(1 - \frac{R^3}{r^3}\right), \tag{2.1.20}$$

where we have converted to spherical coordinates.

The sphere isn't actually grounded – we may need to add charge in order to fix the total charge on the sphere to be zero. With the potential we've worked out so far, the induced charge density on the sphere is

$$\sigma(\theta,\phi) = -\frac{1}{4\pi}\left.\frac{d\phi}{dr}\right|_{r=R} = \frac{3}{4\pi}E_0\cos\theta. \tag{2.1.21}$$

Integrating this, we find a total charge of zero, so in fact our potential and charge density are already correct.

The solution to the previous problem is somewhat non-traditional. A more general solution would look at the Laplace equation and its boundary conditions explicitly, making use of the cylindrical symmetry in the problem. Indeed, if we have a cylindrically symmetric potential $\phi(r,\theta)$, the Laplace equation becomes

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) = 0. \tag{2.1.22}$$

If we substitute the ansatz $\phi(r,\theta) = R(r)\Theta(\theta)$ and separate variables, we find

$$R(r) = Ar^{\ell} + Br^{-(\ell+1)}, \qquad \Theta(\theta) = P_\ell(\cos\theta), \tag{2.1.23}$$

where A and B are constants of integration and P_ℓ is a Legendre polynomial. Since the Laplace equation is linear, the total potential should be a sum of solutions of this type, so we have

$$\phi(r,\theta) = \sum_{\ell=0}^{\infty}\left(A_\ell r^{\ell} + \frac{B_\ell}{r^{\ell+1}}\right)P_\ell(\cos\theta). \tag{2.1.24}$$

Problem 2.4 (J98E3)
An insulated, uncharged, conducting, spherical shell of radius a is placed in a uniform electric field of magnitude E_0. Suppose the shell is cut into two hemispheres at its equator (in the plane perpendicular to the field). What force is required to keep the hemispheres from separating?

This is an identical setup to in the previous problem, but here we will determine the potential using (2.1.24). Since the asymptotic behavior of the potential is $\phi = -E_0 z = -E_0 r\cos\theta$, only the $\ell = 0$ and $\ell = 1$ modes are allowed. We can ignore A_0, and A_1 is clearly $-E_0$, so we have

$$\phi(r,\theta) = -E_0 r\cos\theta + \frac{B_0}{r} + \frac{B_1}{r^2}\cos\theta. \tag{2.1.25}$$

The total charge on the sphere must be zero. Integrating the charge density $\sigma \propto \frac{d\phi}{dr}$, this implies $B_0 = 0$. Furthermore, the field lines should be perpendicular to the sphere at $r = R$, so

$$\left.\frac{d\phi}{d\theta}\right|_{r=R} = \left(E_0 R - \frac{B_1}{R^2}\right)\sin\theta = 0. \tag{2.1.26}$$

This implies $B_1 = E_0 R^3$, so yet again we find

$$\phi(r,\theta) = -E_0 r\cos\theta\left(1 - \frac{R_0^3}{r^3}\right). \tag{2.1.27}$$

The induced charge density is $\frac{3E_0}{4\pi}\cos\theta$, and the z-component of the electric field on the sphere is $3E_0\cos^2\theta$. The total force on one hemisphere is thus

$$F = 2\pi \int_0^\pi \frac{9E_0^2}{4\pi}\cos^3\theta (R^2 \sin\theta\, d\theta) = \frac{9E_0^2 R^2}{8}, \tag{2.1.28}$$

so this is the force required to keep the hemispheres together.

For point charges, we can use (2.1.2) to compute the potential energy. When we deal with continuous charge distributions, we have to generalize this expression to

$$U = \frac{1}{2}\int d\boldsymbol{r}_1 \int d\boldsymbol{r}_2 \frac{\rho(\boldsymbol{r}_1)\rho(\boldsymbol{r}_2)}{|\boldsymbol{r}_2 - \boldsymbol{r}_1|}, \tag{2.1.29}$$

where the factor of $\frac{1}{2}$ comes from double counting terms of the form (2.1.2). This expression is rather unwieldy. We can do better by recognizing (2.1.6), and writing it as

$$U = \frac{1}{2}\int d\boldsymbol{r}\, \rho(\boldsymbol{r})\phi(\boldsymbol{r}). \tag{2.1.30}$$

Better yet, using Gauss's law, this becomes

$$U = \frac{1}{8\pi}\int d\boldsymbol{r}\, |\boldsymbol{E}(\boldsymbol{r})|^2. \tag{2.1.31}$$

We can thus think of the electric field itself as carrying an energy density $\frac{E^2}{8\pi}$.

Problem 2.5 (J07E1)
A point charge Q is located at a distance r away from the center of a thin spherical conducting shell of radius a, which has a net charge also equal to Q. Let $U(r)$ be the total electrostatic potential energy of this system.

a) What is $U(0) - U(\infty)$?

b) Determine the leading behavior of $U(r) - U(\infty)$ as $r \to a$, and make a qualitatively correct sketch showing its important features over the whole range $0 \le r < \infty$.

c) As $r \to \infty$, $U(r) - U(\infty) \to Q^2/r$. Obtain the leading correction to this behavior for large r.[2]

d) If you have not already done so, give the explicit function $U(r) - U(\infty)$ for all r.

[2]The prelim exams are almost always agnostic about Gaussian vs. stupid units. In this case, however, an unfortunate choice was made. It is corrected here.

In the several charge-and-sphere problems we have solved, we have found that the image charge will induce an image charge which will lead to a surface charge distribution on the sphere, and any additional charge is added uniformly. The image charge will be $-Q\frac{a}{r}$ and it will be at a distance $\frac{a^2}{r}$ from the center of the sphere, so we have

$$\phi(s,\theta) = \frac{Q}{\sqrt{s^2+r^2-2rs\cos\theta}} - \frac{Q}{\sqrt{(sr/a)^2+a^2-2rs\cos\theta}}. \tag{2.1.32}$$

This is valid for $s < a$ if $r < a$, or for $s > a$ if $r > a$. In either case we have to add the potential due to the excess charge. The charge distribution on the sphere due to the image charge totals $-Q$ for $r < a$ and $-Qa/r$ for $r > a$, so the excess charge is $Q\,(1 + \min(1, a/r))$. It gives rise to an additional potential

$$\phi_{\text{excess}}(s,\theta) = Q\,(1+\min(1,a/r)) \begin{cases} 1/s & s > a \\ 1/a & s \le a \end{cases}. \tag{2.1.33}$$

We now need to compute the potential energy $U(r)$ (we can ignore $U(\infty) = 0$). Integrating the electric field sounds like pulling out one's own teeth, so we use (2.1.30). The total charge Q has some nonuniform distribution on the sphere, but no matter, the sphere is at constant potential $\phi(a) = \frac{Q}{a}\,(1+\min(1,a/r))$, so this contributes $U_{\text{excess}}(r) = \frac{Q^2}{2a}\,(1+\min(1,a/r))$. The energy of the point charge is

$$U_{\text{point}}(r) = -\frac{Q^2}{2r\sqrt{r^2/a^2+a^2/r^2-2}} + \frac{Q^2}{2}\begin{cases} 1/r(1+a/r) & r > a \\ 2/a & r < a \end{cases}. \tag{2.1.34}$$

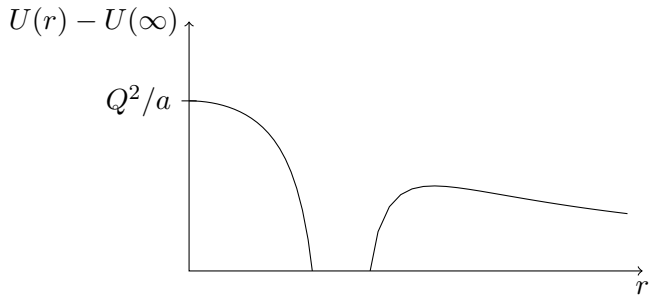
In total and with a bit of simplification, we find

$$U(r) = \frac{1}{2}Q^2\left(-\frac{1}{r|r/a-a/r|} + \begin{cases} \frac{1}{a}(1+a/r)^2 & r > a \\ 4/a & r < a \end{cases}\right). \tag{2.1.35}$$

From this we can read off the answers. We have $U(\infty) = \frac{Q^2}{a}$, and so $U(0) - U(\infty) = \frac{Q^2}{a}$. As $r \to a$, the middle term dominates and we find $U(r) - U(\infty) \sim -\frac{Q^2}{2a}\left|\frac{r-a}{a}\right|^{-1}$. As $r \to \infty$, we have

$$U(r) - U(\infty) = \frac{Q^2}{r} + \frac{Q^2 a}{2r^2}\left(1 - \frac{1}{1-a^2/r^2}\right), \tag{2.1.36}$$

so the leading order correction is $-\frac{Q^2a^3}{2r^4}$. The behavior of the function is plotted below.



2.2 Magnetostatics

All physical quantities need friends, and the electric field is no exception. It has a big job to do, after all – imagine if you were responsible for holding every molecule in the universe together, transporting energy around the globe, performing every digital calculation ever, *and* you had to make people's hair stand up at the science museum? We can hardly ask one field to do all that without someone to keep it company.

The companion is of course the magnetic field, and it really is joined at the hip to the electric field. Indeed, there's no consistent sense in which the electric and magnetic fields are separate objects at all; they represent different projections of a single field strength tensor. Only in the nonrelativistic limit does it make any modicum of sense to talk about them separately. But, since that's the limit where we live, that's how we'll do things.

To pay lip service to relativity, we can give a quick motivation of the magnetic field from Lorentz invariance. Consider a wire carrying current j, and a charge q a distance r from the wire moving parallel to it at velocity v. What force would the charge feel? Electrostatics would tell us zero, because the wire has no net charge density; but the charge is moving around, so electrostatics is pretty useless here. Let's fix this by moving to the rest frame of the charge, by making a Lorentz boost.

Ooh, a Lorentz boost. Recall this means that we take

$$x' = \gamma(x - \beta ct), \qquad ct' = \gamma(ct - \beta x), \tag{2.2.1}$$

where x is the coordinate along the wire, $\beta = v/c$, and $\gamma = (1 - \beta^2)^{-1/2}$. In particular, $x' = \text{const}$ for the charge, so we are indeed in its rest frame. We need to figure out what the wire looks like in this frame. The current j should be thought of as the product of some charge density and a velocity, so if we upgrade this to a four-vector, we have

$$J^\mu = \rho U^\mu = (\rho c, \boldsymbol{j}), \tag{2.2.2}$$

where U^μ is the four-velocity. It follows that after the Lorentz boost, the charge density in the wire is

$$\rho' c = -\beta\gamma j. \tag{2.2.3}$$

It's then quite simple to show that the electrostatic force on the charge should be

$$F' = -\frac{2\beta\gamma q j A}{cr}, \tag{2.2.4}$$

where A is the cross-sectional area of the wire.

Cool, so what does this mean for the lab frame? Since the force is perpendicular to the Lorentz boost, it transforms as $F' = \gamma F$. So,

$$F = -\frac{2vqjA}{c^2 r}. \tag{2.2.5}$$

So a charge moving in the same direction as a current will feel an attraction to that current. This is what a nonrelativistic person would call a magnetic force.

We can derive the same result in the lab frame using Ampére's law,

$$\nabla \times \boldsymbol{B} = \frac{4\pi}{c}\boldsymbol{j}. \tag{2.2.6}$$

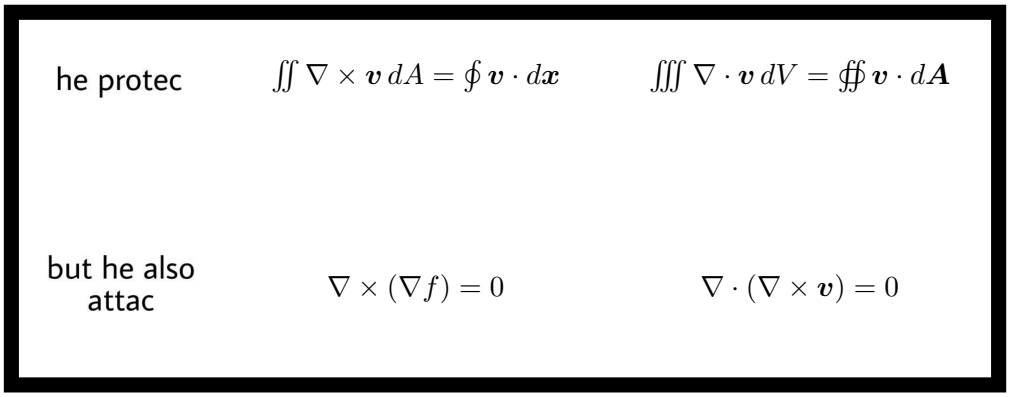


Figure 2.4

By integrating this equation over a cylinder surrounding the wire, we find a magnetic field $\boldsymbol{B} = \frac{2Aj}{cr}$. The electric and magnetic fields act upon particles via the Lorentz force law,

$$\boldsymbol{F} = q\left(\boldsymbol{E} + \frac{\boldsymbol{v}}{c} \times \boldsymbol{B}\right). \tag{2.2.7}$$

For the charge next to the wire, we indeed find $\boldsymbol{F} = -\frac{2vqjA}{c^2 r}\hat{\boldsymbol{r}}$.

When we don't have a nice symmetric current distribution like a wire, we still need some way of solving Ampére's law. Perhaps the easiest way to look at this is by using Gauss's law for magnetism,

$$\nabla \cdot \boldsymbol{B} = 0. \tag{2.2.8}$$

to write

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}. \tag{2.2.9}$$

Figure 2.4 gives a review of why this is the case, and why we have the freedom to replace $\boldsymbol{A}$ with $\boldsymbol{A} + \nabla\chi$ for any function χ. This freedom allows us to fix $\nabla \cdot \boldsymbol{A} = 0$, a convention known as Coulomb gauge, so that we have

$$-\nabla^2 \boldsymbol{A} = \frac{4\pi}{c}\boldsymbol{j}, \tag{2.2.10}$$

which has the form of Poisson's equation in each component. We can immediately integrate to obtain

$$\boldsymbol{A}(\boldsymbol{r}) = -\frac{1}{c}\int d^3\boldsymbol{r}' \frac{\boldsymbol{j}}{|\boldsymbol{r} - \boldsymbol{r}'|}, \tag{2.2.11}$$

and then take the curl to arrive at the Biot-Savart law,

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{1}{c}\int d^3\boldsymbol{r}' \frac{\boldsymbol{j} \times (\boldsymbol{r} - \boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|^3}. \tag{2.2.12}$$

Problem 2.6 (J15E3)
This problem explores some elements of a mass spectrometer. Parts (a) and (b) may be answered independently and treated non-relativistically.

a) An ion of charge $+q$ and mass m is accelerated through a potential V_0 as shown in Fig. 2.5. It enters a region between two very long (in the direction perpendicular to the page) cylindrical electrodes of radius a and b respectively. Find the potentials $V(a)$ and $V(b)$ such that the ion moves in a circle of radius r_0.

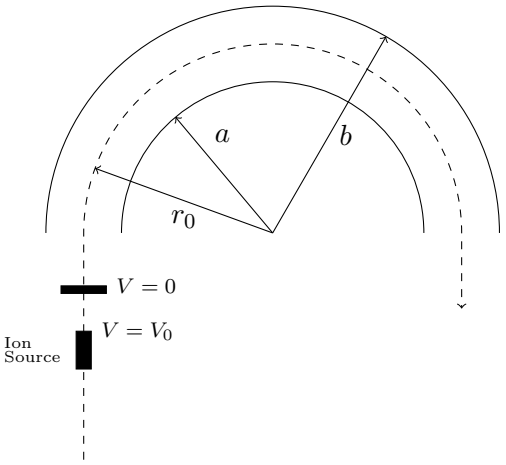


Figure 2.5: Electrostatic filter

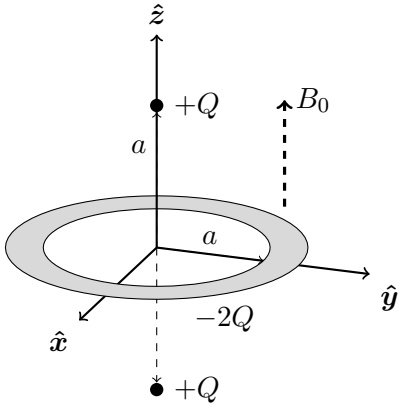


Figure 2.6: Penning trap.

b) A Penning trap is used in Fourier-transform mass spectrometry. At the simplest level, the Penning trap consists of a uniform magnetic field $B_0\hat{\boldsymbol{z}}$ and a quadrupole electric field. Let us assume that the electric field is generated by two positive charges $+Q > 0$ located at $\pm a\hat{\boldsymbol{z}}$ and a uniformly charged ring of radius a and charge $-2Q$ centered around the origin in the xy-plane, as shown in Fig. 2.6. This setup is rotationally symmetric around the z-axis.

i. Close to the origin, the electric field takes the form

$$\boldsymbol{E} = k_z z\hat{\boldsymbol{z}} + k_r r\hat{\boldsymbol{r}} \tag{2.2.13}$$

where $r = \sqrt{x^2 + y^2}$. Determine k_z and k_r.

While the general motion of an ion of mass m and charge $q > 0$ in the Penning trap is quite complicated, here we investigate only two particular cases:

ii. Find the frequency ω_z of small oscillations around the origin in the case where the ion moves only along the z-axis.

iii. Assume the ion moves uniformly along a circle of radius $R \ll a$ in the plane at $z = 0$. What is the angular frequency for this motion? Interpret the answer in the limit of large B_0.

The electric field between the two electrodes goes as $E = \frac{E_0 a}{r}\hat{\boldsymbol{r}}$, so

$$V(b) - V(a) = E_0 a \log \frac{b}{a}. \tag{2.2.14}$$

The ion will have speed $\sqrt{\frac{2V_0q}{m}}$, so if it moves in a circle of radius r_0, the centripetal force on it must be $\frac{2V_0q}{r_0}$. Thus,

$$E_0 = -\frac{2V_0q}{a}, \tag{2.2.15}$$

and it follows that $V(b) - V(a) = -2V_0q \log \frac{b}{a}$.

For the Penning trap, first consider the point $(0, 0, z)$, where z is small. The field due to the point charges is

$$-\frac{Q}{(a-z)^2} + \frac{Q}{(a+z)^2} \approx -\frac{4Q}{a^3}z, \tag{2.2.16}$$

and the field due to the ring is

$$-\frac{2Qz}{(a^2+z^2)^{3/2}} \approx -\frac{2Q}{a^3}z, \tag{2.2.17}$$

so $k_z = -\frac{6Q}{a^3}$. Now we consider the point $(r, 0, 0)$, where r is small. The field due to the point charges is

$$\frac{2Qr}{(a^2+r^2)^{3/2}} \approx \frac{2Q}{a^3}r, \tag{2.2.18}$$

and the field due to the ring is

$$-\frac{2Q}{2\pi a}\int_0^{2\pi} \frac{(-\cos\theta)(a\,d\theta)}{a^2+r^2-2ar\cos\theta} \approx \frac{2Q}{a^3}r, \tag{2.2.19}$$

so $k_r = \frac{4Q}{a^3}$.

If the ion moves only along the z axis, then $\boldsymbol{v}\times\boldsymbol{B} = 0$, so we only have to consider the electrostatic force. The frequency will be

$$\omega_z = \sqrt{-\frac{qk_z}{m}} = \sqrt{\frac{6Qq}{ma^3}}. \tag{2.2.20}$$

Let the angular frequency for rotation in the circle of radius R be ω. Then the speed of the ion is ωR, and the magnetic force upon it will be $\frac{q\omega RB_0}{c}\hat{\boldsymbol{r}}$. This balances the electrostatic force, and in total we need to match the centripetal force, so

$$\frac{4Q}{a^3}R + \frac{q\omega RB_0}{c} = -m\omega^2 R, \tag{2.2.21}$$

implying

$$\omega = -\frac{qB_0}{2mc}\left(1 \pm \sqrt{1 - \frac{16mc^2Q}{q^2B_0^2a^3}}\right). \tag{2.2.22}$$

For large B_0, this becomes

$$\omega \approx -\frac{qB_0}{mc} \qquad \text{or} \qquad \omega \approx -\frac{4Qc}{qB_0a^3}. \tag{2.2.23}$$

The first solution is the usual cyclotron frequency. In the second solution, the magnetic force exactly cancels the electric force to first order, so that the ion can rotate very slowly.

Problem 2.7 (J11E3)
A right circular cylinder of radius R and length L is carrying a uniform current I parallel to the axis of the cylinder.

a) What is the direction and magnitude of the magnetic field inside the cylinder? (Ignore end effects, and other sources of the B field).

b) Next, directed towards the above current-carrying cylinder and parallel to its axis is a parallel monochromatic beam of energetic charged particles. Show that within the following approximation the beam will be focused at a point after passing through the cylinder. Derive an expression for the focal length.

In the derivation neglect scattering and slowing down of the beam's particles due to interactions with the material within the cylinder (other than through the field described above), and make the thin lens approximation by: i) assuming that the cylinder is short compared to the focal length, yet at the same time, ii) ignoring end effects.

c) Consider using the magnet to collect into a parallel beam antiprotons produced by a beam of high-energy protons that strike a target placed at the focal point of the magnetic lens. Specifically: assume the magnet is a cylinder of lithium metal of length 15 cm and radius 1 cm, and the total current it carries is I. What current would be required to collect antiprotons that are produced with a momentum of 10 GeV/c at angles up to 50 mrad relative the beam axis?

We use Ampére's law to derive the magnetic field,

$$\boldsymbol{B} = \frac{2Ir}{cR^2}\hat{\boldsymbol{\theta}}. \tag{2.2.24}$$

In the thin lens approximation, the force $\boldsymbol{F} = -\frac{2Irvq}{c^2R^2}\hat{\boldsymbol{r}}$ will act for a time $t = L/v$ to produce a radial velocity $v_r = -\frac{2ILq}{mc^2R^2}r$. The focal length is thus

$$x = v\frac{r}{-v_r} = \frac{mc^2R^2v}{2ILq}. \tag{2.2.25}$$

And now we confront numbers. Substituting the given parameters, we find

$$x = \frac{1}{I}\frac{(10\,\text{GeV})(1\,\text{cm})^2(3\times10^{10}\,\text{cm/s})}{2(15\,\text{cm})(-e)}, \tag{2.2.26}$$

where e is the elementary charge. Since we want to collect particles at angles up to 50 mrad, we need $x \leq \frac{1\,\text{cm}}{50\,\text{mrad}} = 20\,\text{cm}$. This means

$$|I| \geq \frac{(10\,\text{GeV})(1\,\text{cm})^2(3\times10^{10}\,\text{cm/s})}{2(20\,\text{cm})(15\,\text{cm})(e)} = 5.5\times10^5\,\text{A}, \tag{2.2.27}$$

where we have converted to SI units for reporting a number, since nobody uses statamperes.

Problem 2.8 (J13E1)a) Let a current I circulate in a square of wire of side d lying in the xy plane, with center at the origin. What is the vector potential $\boldsymbol{A}$ at position x_0, where $x_0 \gg d$?

b) What is the magnetic field $\boldsymbol{B}$ at x_0?

c) At x_0 lies a charge q_0 at rest. Calculate the force acting on the charge, and the force acting on the loop.

d) Now boost to a frame where the charge q_0 and the loop are both moving with speed $+v_0\hat{\boldsymbol{x}}$. What is the electric field $\boldsymbol{E}'$ due to the loop acting on the charge?

e) What is the total force $\boldsymbol{F}_{\text{tot}}$ acting on the charge q_0 in this frame?

To determine the vector potential due to the current loop, we integrate Poisson's equation. The horizontal portions of the current clearly cancel, and we obtain

$$\boldsymbol{A}(x_0) = \frac{I}{c}\hat{\boldsymbol{y}} \int_{-d/2}^{d/2} dy \left(\frac{1}{\sqrt{(x_0 - d/2)^2 + y^2}} - \frac{1}{\sqrt{(x_0 + d/2)^2 + y^2}} \right) \approx \frac{Id}{cx_0^2}\hat{\boldsymbol{y}}. \tag{2.2.28}$$

There will be azimuthal symmetry at large x_0, so really we have $\boldsymbol{A} = \frac{Id}{cr^2}\hat{\boldsymbol{\theta}}$. The magnetic field is then

$$\hat{\boldsymbol{B}} = \frac{1}{r}\frac{\partial}{\partial r}\left(r \cdot \frac{Id}{cr^2} \right) \hat{\boldsymbol{z}} = -\frac{Id}{cr^2}\hat{\boldsymbol{z}}. \tag{2.2.29}$$

A charge q_0 at rest will feel no force, because there is only a magnetic field. If we boost with $\boldsymbol{\beta} = -\frac{v_0}{c}\hat{\boldsymbol{x}}$, then we have

$$\boldsymbol{E} = \gamma\beta B_z \hat{\boldsymbol{y}} = \gamma \frac{Idv_0}{c^2 r^2}\hat{\boldsymbol{y}}. \tag{2.2.30}$$

Furthermore, the magnetic field will become $\boldsymbol{B}' = \gamma\frac{Id}{cr^2}\hat{\boldsymbol{z}}$. Thus, the total force on the charge is

$$\boldsymbol{F}' = q_0 \left(\boldsymbol{E}' + \frac{v_0\hat{\boldsymbol{x}}}{c} \times \boldsymbol{B}' \right) \tag{2.2.31}$$

$$= \gamma q_0 \left(\frac{Id}{cr^2}\frac{v_0}{c}\hat{\boldsymbol{y}} + \frac{Id}{cr^2}\frac{v_0}{c}\hat{\boldsymbol{x}} \times \hat{\boldsymbol{z}} \right) \tag{2.2.32}$$

$$= 0, \tag{2.2.33}$$

a reassuring result.

Problem 2.9 (M02E3)
A current I, carried by freely moving electrons, runs through a cylindrical wire with radius r_0. Assuming that the electrons are moving with velocity v and that the total charge per unit length in the wire vanishes (in the lab frame), find the radial profile of the current. What is the voltage difference between the center and the edge of the cylinder?

Let the number density of electrons be $n_e(r)$, and the number density of (stationary) protons be n_p. We will pretend that electrons carry positive unit charge, since such a rescaling of charge will not affect any physical conclusions. Then the charge and current distributions will be

$$\rho(r) = n_e(r) - n_p, \qquad \boldsymbol{j}(r) = n_e(r) v \hat{\boldsymbol{z}}. \tag{2.2.34}$$

The electric field will be

$$\boldsymbol{E}(r) = \frac{2\hat{\boldsymbol{r}}}{r} \int_0^r (n_e(r) - n_p)(2\pi r\, dr), \tag{2.2.35}$$

and the magnetic field will be

$$\boldsymbol{B}(r) = \frac{2\hat{\boldsymbol{\theta}}}{cr} \int_0^r n_e(r) v (2\pi r\, dr). \tag{2.2.36}$$

In order to have a stable current distribution, the total force on electrons should vanish, so

$$\boldsymbol{E} + \frac{v\hat{\boldsymbol{z}}}{c} \times \boldsymbol{B} = \frac{2\hat{\boldsymbol{r}}}{r} \int_0^r (n_e(r) - n_p)(2\pi r\, dr) - \frac{2v\hat{\boldsymbol{r}}}{c^2 r} \int_0^r n_e(r) v (2\pi r\, dr) = 0. \tag{2.2.37}$$

Multiplying by r and taking the derivative with respect to r, we find

$$n_e(r) = \frac{n_p}{1 - v^2/c^2}, \tag{2.2.38}$$

and in particular the distribution of charge carriers is independent of radius.

This is clearly inconsistent with the assumption that the total charge per unit length vanishes in the lab frame. However, noting the factor of γ^2 which appeared, we might look instead at the frame of the electrons. In the lab frame we have

$$\rho = n_p(\gamma^2 - 1) = n_p \gamma^2 \beta^2, \qquad j_z = n_p \gamma^2 v. \tag{2.2.39}$$

Upon boosting by $v\hat{\boldsymbol{z}}$, we obtain

$$\rho' = \gamma \left(\rho - \beta \frac{j_z}{c} \right) = 0, \qquad j_z' = \gamma(j_z - v\rho) = n_p \gamma v, \tag{2.2.40}$$

so this is the frame where the charge per unit length vanishes.

In any case, the voltage difference between the center and edge of the cylinder, in the lab frame, is

$$\delta\phi = \int_0^r \boldsymbol{E}(r) \cdot \hat{\boldsymbol{r}}\, dr = 2\pi n_p \gamma^2 \beta^2 r_0^2. \tag{2.2.41}$$

In electrostatics, the simplest setup we could have was a charge sitting in some fixed potential. The potential energy of the charge is given by $q\phi$. There is no analogue of charge in the magnetic case, owing to the equation $\nabla \cdot \boldsymbol{B} = 0$. The simplest source of magnetic fields is a current, which will turn out to be more similar to an electric dipole.

For the sake of comparison, take an electric dipole formed by charges q and $-q$, and let the vector

from the negative to the positive charge be $\boldsymbol{a}$. The potential energy of this configuration will be

$$U = q\phi(\boldsymbol{x} + \boldsymbol{a}) - q\phi(\boldsymbol{x}), \tag{2.2.42}$$

where $\boldsymbol{x}$ is the position of the negative charge. If we let $\boldsymbol{a}$ shrink to zero while holding $q\boldsymbol{a}$ fixed, we find

$$U = q\boldsymbol{a} \cdot \nabla\phi = -(q\boldsymbol{a}) \cdot \boldsymbol{E}. \tag{2.2.43}$$

This shows that an electric dipole wants to align itself with the electric field, which makes sense.

A magnetic dipole can be formed by taking a current loop. For the sake of concreteness, imagine a circular loop of radius a carrying a current I, with a normal vector $\hat{\boldsymbol{n}}$. If we take the loop small enough, the net magnetic force will vanish, just as the net electric force on an electric dipole vanishes. There will, however, be a nonzero torque which we can compute. It is

$$\boldsymbol{\tau} = \int_0^{2\pi} d\theta \, (a\hat{\boldsymbol{r}}) \times (Ia(\hat{\boldsymbol{n}} \times \hat{\boldsymbol{r}}) \times \boldsymbol{B}) \, . \tag{2.2.44}$$

The Geneva Convention classifies integrals with three or more cross products as war crimes, so we'd better simplify. We have

$$\hat{\boldsymbol{r}} \times ((\hat{\boldsymbol{n}} \times \hat{\boldsymbol{r}}) \times \boldsymbol{B}) = (\hat{\boldsymbol{r}} \cdot \boldsymbol{B})(\hat{\boldsymbol{n}} \times \hat{\boldsymbol{r}}). \tag{2.2.45}$$

Furthermore, $\hat{\boldsymbol{B}} = B\cos\theta\hat{\boldsymbol{n}} + B\sin\theta\hat{\boldsymbol{x}}$, where $\hat{\boldsymbol{x}}$ is some direction in the plane of the loop and θ is the angle between $\boldsymbol{B}$ and $\hat{\boldsymbol{n}}$. Using this representation, we find

$$\boldsymbol{\tau} = \pi I a^2 B \sin\theta\hat{\boldsymbol{x}}. \tag{2.2.46}$$

This torque could be derived from the potential energy

$$U = -(\pi a^2 I\hat{\boldsymbol{n}}) \cdot \boldsymbol{B} = -(IA\hat{\boldsymbol{n}}) \cdot \boldsymbol{B}. \tag{2.2.47}$$

By analogy with the electrostatic case, we define a magnetic moment $\boldsymbol{m} = IA\hat{\boldsymbol{n}}$, so that we have the potential energy $-\boldsymbol{m} \cdot \boldsymbol{B}$ and the dipole wants to align itself with the magnetic field.

Problem 2.10 (J12E3)
An uncharged particle of mass M and magnetic moment m sits in a vacuum above a superconductor. The surface of the superconductor is the infinite plane $z = 0$. It is an ideal superconductor, so the magnetic field vanishes ($\boldsymbol{B} = 0$) inside the superconductor ($z < 0$). The particle's position and the orientation of its magnetic moment are those that minimize its energy in the presence of gravity $\boldsymbol{g} = -g\hat{\boldsymbol{z}}$.

a) How far above the superconductor does the particle sit?

b) What is the orientation of its magnetic moment relative to the z axis?

Clearly we need to use the potential energy $-\boldsymbol{m}\cdot\boldsymbol{B}$ to solve this problem, but first we need the field $\boldsymbol{B}$. We start by determining the field due to a magnetic moment in a vacuum. For a ring of radius a at the origin carrying a current I, oriented with its normal along the z axis, the vector potential will be

$$A_\phi(r,\theta) = \frac{1}{c}\int_0^{2\pi} \frac{Ia\cos\phi\, d\phi}{\sqrt{r^2+a^2-2ra\sin\theta\cos\phi}}. \tag{2.2.48}$$

For $a \ll r$, this reduces to

$$A_\phi(r,\theta) = \frac{I\pi a^2}{cr^2}\sin\theta, \tag{2.2.49}$$

and so

$$\boldsymbol{B}(r,\theta) = \nabla\times\boldsymbol{A} = \frac{m}{cr^3}\left(2\cos\theta\hat{\boldsymbol{r}} + \sin\theta\hat{\boldsymbol{\theta}}\right), \tag{2.2.50}$$

where we have replaced $m = \pi a^2 I$.

If we place a magnetic moment at height h above a superconductor, then somehow the field must be screened within it. We can imagine this happening via some current distribution in the superconductor. This current distribution will only be stable if the normal component of the magnetic field vanishes at the surface of the superconductor.

In order to satisfy this boundary condition, we can employ an image moment with opposite vertical component at height $-h$, directly below the real moment. It is straightforward to show that this makes the magnetic field purely tangential on the surface of the superconductor. The real moment will then feel a magnetic field due to its image. The real moment is at coordinates $(2h, \pi+\theta)$ with respect to the position and orientation of its image, so it perceives a field

$$\boldsymbol{B} = \frac{m}{8ch^3}\left(-2\cos\theta\hat{\boldsymbol{z}} - \sin\theta\hat{\boldsymbol{x}}\right). \tag{2.2.51}$$

This gives a potential energy of

$$U = Mgh + \frac{m^2}{8ch^3}\left(\cos^2\theta + 1\right). \tag{2.2.52}$$

Clearly the moment will be oriented at $\theta = \frac{\pi}{2}$ to minimize this energy. We then have

$$\frac{dU}{dh} = Mg - \frac{3m^2}{8ch^4} = 0 \implies h = \left(\frac{3m^2}{8cMg}\right)^{1/4}. \tag{2.2.53}$$

2.3 Electromagnetic Fields in Matter

We've solved a few problems involving conductors, which screen electric fields. If a material isn't an ideal conductor, meaning its electrons are to some degree bound in atoms, how does it respond to an electric field? Heuristically, each charge carrier in the material will move a little bit in response to the field – protons will move along the field, and electrons will move in the opposite direction. These effects cancel in the bulk, but lead to a nonzero induced surface charge.

This surface charge will, in turn, generate an electric field which opposes the original field. This suggests how we can interpolate between perfect insulators and perfect conductors. In a perfect

insulator there would be no surface charge and no screening of the field; in a perfect conductor, the surface charge would be just enough to exactly cancel the field; for a real material, the situation is something in the middle, with partial screening of the electric field.

It's conventional to make a distinction between these induced surface charges, called bound charges and denoted ρ_b, and other charges, called free charges and denoted ρ_f. Both types of charge are perfectly good, so they both source electric fields:

$$\nabla \cdot \boldsymbol{E} = 4\pi(\rho_f + \rho_b). \tag{2.3.1}$$

As discussed above, this total field will be a superposition of the applied field and the response due to polarization of the material. We could imagine a field $\boldsymbol{D}$, called the electric displacement field, which is sourced only by the free charges:

$$\nabla \cdot \boldsymbol{D} = 4\pi\rho_f. \tag{2.3.2}$$

The rest of the electric field is due to the bound charges.

The relationship between $\boldsymbol{D}$ and $\boldsymbol{E}$ depends on the exact behavior of the material, and so in principle can be very complicated. Typically we ignore the complexity and assume a linear relationship,

$$\boldsymbol{D} = \epsilon\boldsymbol{E}, \tag{2.3.3}$$

where $\epsilon = 1 + 4\pi\chi$ is the relative permittivity, and χ is the susceptibility of the material. We then define a polarization density $\boldsymbol{P}$ by

$$\boldsymbol{P} = \chi_e\boldsymbol{E}, \tag{2.3.4}$$

so that we can write

$$\boldsymbol{D} = \boldsymbol{E} + 4\pi\boldsymbol{P}. \tag{2.3.5}$$

There are two dimensionless constants we can compare in this derivation between SI units and Gaussian units. In SI units, the relative permittivity would be written ϵ/ϵ_0, and it is analogous to the ϵ here; and also, there is a susceptibility χ_e in both unit systems. The relative permittivities do agree, but in SI units it is conventional to write $\epsilon/\epsilon_0 = 1 + \chi$, leading to a discrepancy by a factor of 4π:

$$\chi_e^{\mathrm{G}} = \frac{\chi_e^{\mathrm{SI}}}{4\pi}. \tag{2.3.6}$$

This is important to remember when comparing expressions given in Gaussian units to expressions given in inferior units.

In any case, there is a very similar story to be told for magnetism. Since electrons move in orbitals around atoms, atoms are effectively magnetic dipoles (not to mention the intrinsic magnetic dipole moment of the electron, which we'll discuss in Chapter 3). So, if we apply a magnetic field to a material, the dipoles within it will align with the field.

In reality, it's not nearly this simple. The Bohr-van Leeuwen theorem shows that magnetization of a material can't occur within the framework of classical physics. So, although we'll give a classical description of the phenomenon in what follows, it's truly a quantum effect. The only consequence of this we'll need is that, while materials are (almost) always dielectrics – creating fields which oppose the applied field – they can be either paramagnetic or diamagnetic. Paramagnetic materials have magnetization aligned with the applied field.

We now proceed as before. The total current density is made up of bound and free pieces, $\boldsymbol{j} = \boldsymbol{j}_f + \boldsymbol{j}_b$. We define a field $\boldsymbol{H}$ which satisfies

$$\nabla \times \boldsymbol{H} = \frac{4\pi}{c}\boldsymbol{j}_f, \tag{2.3.7}$$

and assume a linear relationship

$$\boldsymbol{B} = \mu \boldsymbol{H}, \tag{2.3.8}$$

where μ is the relative permeability. Note that this is in some sense the opposite of the convention for electric fields. If we take $\mu = 1 + 4\pi\chi_m$, and define a magnetization

$$\boldsymbol{M} = \chi_m \boldsymbol{H}, \tag{2.3.9}$$

then we have

$$\boldsymbol{B} = \boldsymbol{H} + 4\pi \boldsymbol{M}. \tag{2.3.10}$$

From this equation we see that materials with positive magnetic susceptibility $\chi_m > 0$ will enhance the magnetic field, and so these are the paramagnetic materials. Diamagnetic materials have $\chi_m < 0$.

So much for the fields. This discussion is based on the effects of bound charges and currents, so it would be nice to know what these actually are. It's almost immediate from (2.3.5) and (2.3.10) that

$$\rho_b = -\nabla \cdot \boldsymbol{P}, \qquad \boldsymbol{j}_b = c\nabla \times \boldsymbol{M}. \tag{2.3.11}$$

In principle this is the whole story, but in practice, typically $\boldsymbol{P}$ and $\boldsymbol{M}$ are uniform throughout a material and then zero outside of it, so there will be some singular distribution of charge and current on the surface. To determine the surface charge, we can integrate the bound charge density over a small region around the surface and find a surface charge density

$$\sigma_b = -\boldsymbol{P} \cdot \hat{\boldsymbol{n}}, \tag{2.3.12}$$

where $\hat{\boldsymbol{n}}$ is an outward facing normal. Similarly, we can integrate the bound current density around a small ring at the surface, and find a surface current density

$$\kappa_b = c\boldsymbol{M} \times \hat{\boldsymbol{n}}. \tag{2.3.13}$$

Problem 2.11 (J02E3)
It has proven possible to levitate objects (frogs!) on the surface of the earth in regions of high magnetic field gradients. This problem explores how a "spherical frog" might be levitated above a permanent magnet.

a) Consider a magnetic disk of radius a and thickness $h \ll a$. The magnetic material has a constant magnetic moment/volume $\boldsymbol{M}$ oriented parallel to the axis of the disk, the z axis. Find the magnetic field $B(z)$ along the z axis.

b) The "spherical frog" to be levitated has a radius b and mass k and (relative) diamagnetic permeability μ. Assume that $b \ll a$, so that the magnetic field is roughly constant across the frog. Find the maximum value for the mass k for there to be an equilibrium point above the disk in terms of m, a, b, h, μ, and the position z_0 above the disk where that occurs.

The magnetization induces a bound current

$$\kappa_b = Mc\hat{\boldsymbol{\theta}} \tag{2.3.14}$$

on the edge of the disk. We can use the Biot-Savart law to determine the field due to this current,

$$\boldsymbol{B} = M\int_0^{2\pi} \frac{\hat{\boldsymbol{\theta}}\times(\boldsymbol{r}-\boldsymbol{r'})}{a^2+z^2}(a\,d\theta). \tag{2.3.15}$$

By symmetry, only the $\hat{\boldsymbol{z}}$ component survives, which yields a factor $\frac{a}{\sqrt{a^2+z^2}}$. This gives

$$\boldsymbol{B} = 2(M\pi a^2 h)\frac{1}{(a^2+z^2)^{3/2}}. \tag{2.3.16}$$

We define $m \equiv M\pi a^2 h$, the total magnetic dipole moment of the disk.

We now need to determine the potential energy of the frog in the field. The magnetization of the frog is

$$\boldsymbol{M} = \chi_M \boldsymbol{H} = \frac{\mu-1}{4\pi}\boldsymbol{H}, \tag{2.3.17}$$

and so its total magnetic dipole moment is $\boldsymbol{m} = \frac{\mu-1}{3}b^3\boldsymbol{H}$. This gives a potential energy

$$U = kgz - \boldsymbol{m}\cdot\boldsymbol{B} = kgz - \frac{4}{3}(\mu-1)\frac{b^3m^2}{(a^2+z^2)^3}. \tag{2.3.18}$$

The equilibrium point will occur where

$$0 = \frac{\partial U}{\partial z} = kg + 4(\mu-1)b^3m^2\frac{2z}{(a^2+z^2)^4}. \tag{2.3.19}$$

To find the maximum possible mass, we need to find the maximum possible value of

$$F_{\text{mag}} = 4(1-\mu)b^3m^2\frac{2z}{(a^2+z^2)^4}. \tag{2.3.20}$$

Taking a derivative we find $\frac{dF_{\text{mag}}}{dz} \propto a^2 - 7z^2$, so the maximum will occur at $z = a/\sqrt{7}$, which gives

$$k = \frac{1}{g}F_{\text{mag}} = \sqrt{7}\left(\frac{7}{8}\right)^3(1-\mu)\frac{b^3m^2}{a^7}. \tag{2.3.21}$$

Note that none of this is possible unless $\mu < 1$, i.e., unless the frog is diamagnetic.

Problem 2.12 (M02E2)
A cylinder of radius a and dielectric constant ϵ is placed along the z-axis in a electric field, whose form is $\boldsymbol{E}_i = E_0\hat{\boldsymbol{x}} + E_1[(x/a)\hat{\boldsymbol{x}} - (y/a)\hat{\boldsymbol{y}}]$ before the cylinder is placed in the field. Give expressions for the total electric field $\boldsymbol{E}$, the displacement field $\boldsymbol{D}$ and the polarization density $\boldsymbol{P}$ everywhere.

We start by finding the electric potential. We can write the general expansion in cylindrical coordinates

$$\phi(r,\theta) = \sum_{n=1}^{\infty} \left(\left(\frac{r}{a}\right)^n (a_n \cos n\theta + b_n \sin n\theta) + \left(\frac{r}{a}\right)^{-n} (c_n \cos n\theta + d_n \sin n\theta) \right). \tag{2.3.22}$$

We will need to find the potential in the regions $r < a$ and $r > a$ separately, and so we label the coefficients as $a_n^{\pm}$, etc.

There is no free charge, so $\nabla \cdot \boldsymbol{D} = 0$. This in particular means that the perpendicular component of $\boldsymbol{D}$ is continuous across the boundary of the cylinder. Using $\boldsymbol{D} = \epsilon \boldsymbol{E}$, this gives the jump condition

$$\epsilon \left.\frac{d\phi}{dr}\right|_{a-\delta} = \left.\frac{d\phi}{dr}\right|_{a+\delta}. \tag{2.3.23}$$

Additionally, continuity requires $\phi(a-\delta) = \phi(a+\delta)$. And finally, agreement with the unperturbed field at $r \gg a$ requires

$$\frac{\partial \phi}{\partial r} \to -\boldsymbol{E}_i \cdot \hat{\boldsymbol{r}} = -E_0 \cos\theta - E_1 \cos 2\theta. \tag{2.3.24}$$

Clearly we will only need the $n = 1$ and $n = 2$ modes and $b_n = d_n = 0$, so we just need to solve for the eight coefficients $a_{1,2}^{\pm}$ and $c_{1,2}^{\pm}$. We must set $c_1^- = c_2^- = 0$ to avoid a singularity at the origin. Continuity at the boundary implies

$$a_1^+ + c_1^+ = a_1^-, \qquad a_2^+ + c_2^+ = a_2^-. \tag{2.3.25}$$

The surface charge condition implies

$$\frac{a_1^+ - c_1^+}{a}\cos\theta + 2\frac{a_2^+ - c_2^+}{a}\cos 2\theta = \epsilon\frac{a_1^-}{a}\cos\theta + 2\frac{a_2^-}{a}\cos 2\theta. \tag{2.3.26}$$

The boundary condition at infinity fixes

$$a_1^+ = -aE_0, \qquad a_2^+ = -\frac{a}{2}E_1. \tag{2.3.27}$$

This is enough information to solve for all the coefficients. We find

$$\begin{aligned} c_1^+ &= a\frac{\epsilon-1}{\epsilon+1}E_0, \quad c_2^+ = \frac{a}{2}\frac{\epsilon-1}{\epsilon+1}E_1, \\ a_1^- &= -\frac{2a}{\epsilon+1}E_0 \quad a_2^- = -\frac{a}{\epsilon+1}E_1. \end{aligned} \tag{2.3.28}$$

From this we have

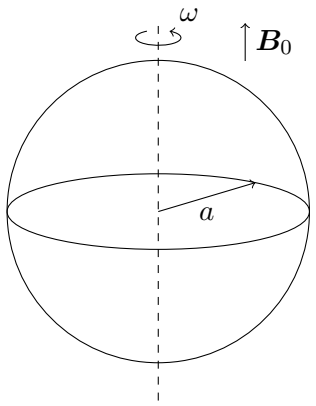
$$\boldsymbol{E} = \begin{cases} \left(E_0\left(1 + \dfrac{a^2}{r^2}\dfrac{\epsilon-1}{\epsilon+1}\right)\cos\theta + E_1\left(\dfrac{r}{a} + \dfrac{a^3}{r^3}\dfrac{\epsilon-1}{\epsilon+1}\right)\cos 2\theta\right)\hat{r} \\ -\left(E_0\left(1 + \dfrac{a^2}{r^2}\dfrac{\epsilon-1}{\epsilon+1}\right)\sin\theta + E_1\left(\dfrac{r}{a} + \dfrac{a^3}{r^3}\dfrac{\epsilon-1}{\epsilon+1}\right)\sin 2\theta\right)\hat{\theta} & r > a \\ \dfrac{2}{\epsilon+1}\left[\left(E_0\cos\theta + E_1\dfrac{r}{a}\cos 2\theta\right)\hat{r} - \left(E_0\sin\theta + E_1\dfrac{r}{a}\sin 2\theta\right)\hat{\theta}\right] & r < a \end{cases} \tag{2.3.29}$$

Note that if we take $\epsilon = 1$, we recover $\boldsymbol{E} = \boldsymbol{E}_i$ everywhere, a reassuring check.

The displacement field $\boldsymbol{D}$ differs from this only by a factor of ϵ in the $r < a$ field. The polarization $\boldsymbol{P}$ vanishes for $r < a$, and is given by $\frac{\epsilon-1}{4\pi}\boldsymbol{E}$ for $r < a$. Neither of these are worth writing out in all their ugly glory.

Problem 2.13 (J10E2)
A solid metallic sphere of radius a has finite conductivity, carries no net electric charge, and is free to rotate without friction about a vertical axis through its center. The region outside the sphere is vacuum. There is a uniform magnetic field with flux density $\boldsymbol{B}_0$ parallel to the axis.

The sphere is given an impulse that starts it spinning around the axis and there is some initial Ohmic dissipation. After the dissipation has ceased, the sphere is in a steady state of rigid rotation with constant angular velocity ω_∞.



In steady state, to lowest order in both B_0 and ω_∞ find:

a) The electric field $\boldsymbol{E}(\boldsymbol{r})$ and electric potential $\phi(\boldsymbol{r})$ in the interior of the sphere, $r < a$. (Give these in the non-rotating "laboratory frame.")

b) The electric potential outside the sphere. (Express your answer in spherical coordinates (r, θ, φ).) State the nature of the electric field it describes (i.e., monopole, dipole, quadrupole, etc.).

c) The induced bulk and surface charge density distributions in the conductor that give rise to this potential.

The force on a charge at radius $r < a$ is

$$\boldsymbol{F} = q\boldsymbol{E} + \frac{q\omega r \sin\theta}{c} B_0(\hat{\boldsymbol{\varphi}} \times \hat{\boldsymbol{z}}) = q\boldsymbol{E} + \frac{q\omega r B_0}{c}\left(\frac{1}{2}\sin 2\theta\hat{\boldsymbol{\theta}} + \sin^2\theta\hat{\boldsymbol{r}}\right). \tag{2.3.30}$$

Therefore, we must have

$$\boldsymbol{E} = -\frac{\omega r B_0}{c}\left(\frac{1}{2}\sin 2\theta\hat{\boldsymbol{\theta}} + \sin^2\theta\hat{\boldsymbol{r}}\right) \tag{2.3.31}$$

in the interior of the sphere. This field comes from the potential

$$\phi = \frac{\omega B_0}{2c} r^2 \sin^2\theta - \frac{\omega B_0 a^2}{3c}, \tag{2.3.32}$$

where we have added a constant term to make the average value of ϕ on the sphere vanish.

We need to smoothly connect this to a potential outside the sphere which solves Laplace's equation and decays to zero. Using the expansion in spherical coordinates,

$$\phi(r,\theta,\varphi) = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}} \right) P_\ell(\cos\theta), \tag{2.3.33}$$

we see that only the $\ell = 2$ term contributes (by construction). Outside the sphere, we have

$$\phi(r,\theta,\varphi) = \frac{\omega B_0 a^5}{6cr^3} \left(1 - 3\cos^2\theta\right). \tag{2.3.34}$$

This is the field of a quadrupole since it comes from the $\ell = 2$ term.

The induced charge density in the conductor is

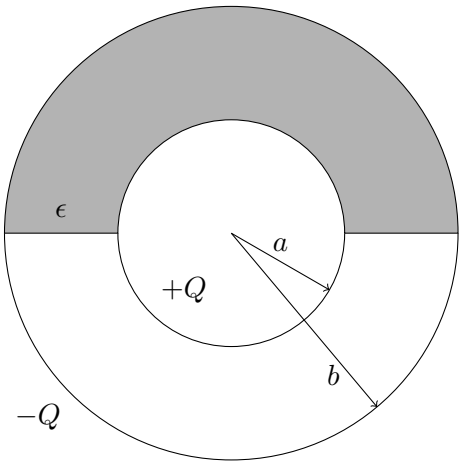
$$\rho_b = \frac{1}{4\pi}\nabla\cdot\boldsymbol{E} = -\frac{3\omega B_0}{4\pi c}\sin^2\theta - \frac{\omega B_0}{4\pi c}\frac{1+3\cos 2\theta}{2} = -\frac{\omega B_0}{2\pi c}, \tag{2.3.35}$$

and the induced surface charge density is

$$\sigma_b = -\frac{1}{4\pi}\left(\left.\frac{\partial\phi}{\partial r}\right|_{a+\delta} - \left.\frac{\partial\phi}{\partial r}\right|_{a-\delta} \right) = -\frac{5\omega B_0 a}{8\pi c}\sin^2\theta + \frac{\omega B_0 a}{4\pi c}. \tag{2.3.36}$$

It is straightforward to check that the total induced charge vanishes.

Problem 2.14 (J05E1)
Two concentric conducting spheres of radii a and b carry charges $+Q$ and $-Q$ as shown. The radial gap between the spheres is half filled with a material of dielectric constant ϵ and half filled with vacuum.



a) Find the electric field $\boldsymbol{E}$ and the displacement field $\boldsymbol{D}$ everywhere between the spheres.

b) What is the bound charge density on the surfaces of the dielectric?

The displacement field $\boldsymbol{D}$ is sourced only by the free charge, so for $a < r < b$ we have

$$\boldsymbol{D} = \frac{Q}{r^2}\hat{\boldsymbol{r}}. \tag{2.3.37}$$

It follows that

$$\boldsymbol{E} = \begin{cases} \frac{Q}{\epsilon r^2}\hat{\boldsymbol{r}} & \theta < \pi/2 \\ \frac{Q}{r^2}\hat{\boldsymbol{r}} & \theta > \pi/2 \end{cases}. \tag{2.3.38}$$

The polarization is

$$\boldsymbol{P} = \frac{\epsilon - 1}{4\pi\epsilon}\frac{Q}{r^2}\hat{\boldsymbol{r}} \qquad a < r < b, \quad \theta < \pi/2. \tag{2.3.39}$$

Thus, the bound charge density is

$$\sigma_{\text{out}} = \frac{\epsilon - 1}{4\pi\epsilon}\frac{Q}{b^2}, \qquad \sigma_{\text{in}} = -\frac{\epsilon - 1}{4\pi\epsilon}\frac{Q}{a^2}. \tag{2.3.40}$$

Conductors also have interesting properties beyond their equilibrium states (which we have already discussed). The entire field of magnetohydrodynamics is devoted to studying conducting fluids, but that's hard, so nevermind. Conductors respond to electric fields with currents, via

$$\boldsymbol{j} = \sigma\boldsymbol{E}. \tag{2.3.41}$$

The constant σ is the conductivity. It can be a tensor for nonisotropic materials, but usually we ignore this and treat it as a scalar.

We may also define a resistivity $\rho = \sigma^{-1}$, such that

$$\boldsymbol{E} = \rho\boldsymbol{j}. \tag{2.3.42}$$

If the conductor is a wire with length L and cross sectional area A, then $|\boldsymbol{E}| = V/L$ and $|\boldsymbol{j}| = I/A$, so we have

$$V = \frac{\rho L}{A}I \equiv IR, \tag{2.3.43}$$

where we have defined the bulk resistance $R = \frac{\rho L}{A}$. Both this and (2.3.41) are known as Ohm's law.

Problem 2.15 (J11E2)
A pulsed beam of charged particles is shot into a finite electrically isolated plate of ohmic conductance σ and dielectric coefficient ϵ. At the end of the pulse (at time $t = 0$) the charge per unit volume in the plate is non-uniform and given at $\boldsymbol{r}$ by $\rho_0(\boldsymbol{r})$, where the position vector $\boldsymbol{r}$ specifies points inside the plate. You may neglect any magnetic fields in the plate.

a) Show that the final state of static equilibrium is one in which the charge is deposited only on the surface of the plate.

b) Find the equation governing the charge distribution $\rho_0(\boldsymbol{r})$ for $t > 0$ as the system approaches static equilibrium.

c) Solve this equation and show that the interior charge moves to the surface with a characteristic time constant τ. Determine the expression for τ.

If there is charge in the interior of the plate, pick some point $\boldsymbol{r}$ at which $\rho_{\text{eq}}(\boldsymbol{r}) \neq 0$, and draw a sphere around it, small enough that the charge in the sphere is roughly uniform. Then Gauss's law tells us that there is force on the charge on the boundary of the sphere, so this is not an equilibrium state.

The continuity equation for charge is

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \boldsymbol{j} = 0. \tag{2.3.44}$$

Using Ohm's law, we find

$$\frac{\partial \rho}{\partial t} + 4\pi\sigma\rho = 0. \tag{2.3.45}$$

The solution to this equation is

$$\rho_t(\boldsymbol{r}) = \rho_0(\boldsymbol{r}) e^{-4\pi\sigma t}, \tag{2.3.46}$$

so the characteristic time is $\tau = \frac{1}{4\pi\sigma}$.

2.4 Electrodynamics

Fields can change, and when fields change they change other fields. It's an ambitious crossover event. Faraday discovered that a changing magnetic field gives rise to a circulating electric field, via

$$\nabla \times \boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t}. \tag{2.4.1}$$

This is fittingly called Faraday's law. Note that this means we can't necessarily write $\boldsymbol{E} = -\nabla\phi$ anymore. Instead we need the more general relation,

$$\boldsymbol{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t}, \tag{2.4.2}$$

which is tailor-made to be consistent with Faraday's law.

Problem 2.16 (J15E1)
A small wire loop of radius a lies in the xy-plane, centered on the origin. A magnetic moment $\boldsymbol{m} = m\hat{\boldsymbol{z}}$ travels up along the z axis with constant speed v. It passes through the center of the wire loop at $t = 0$.

a) Compute the emf $\mathcal{E}(t)$ around the loop.

b) If the loop has resistance R, find the Joule heat $P(t)$. Assume the loop is fixed in position.

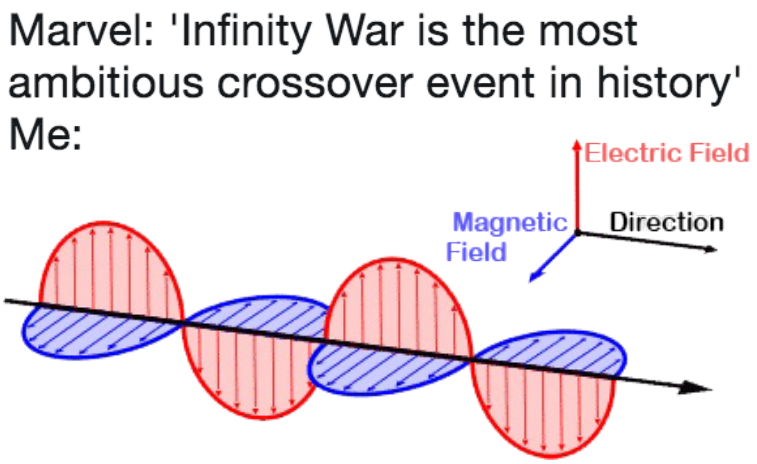


Figure 2.7

c) Now consider the case where a uniform linear charge density λ is glued to a nonconducting loop (same orientation and radius as above), and the loop is allowed to spin. What is the position of m at the time the loop attains its largest angular momentum, $L_{\max}$? What is the value of $L_{\max}$? Assume the dipole began its constant-velocity motion at $t = -\infty$, and that the loop was at rest then.

The emf is given by the integral of $\boldsymbol{E}$ around the loop. We can obtain this by integrating Faraday's law over a surface bounded by the loop, for which we take a spherical cap. When the magnetic moment is at $-z$, the cap spans angles $\theta \in [0, \tan^{-1}(a/z)]$ and has radius $\sqrt{z^2+a^2}$. The magnetic field of a dipole $\boldsymbol{m}$ at the origin is

$$\boldsymbol{B} = \frac{m}{cr^3}\left(2\cos\theta\hat{\boldsymbol{r}} + \sin\theta\hat{\boldsymbol{\theta}}\right). \tag{2.4.3}$$

Thus,

$$\oiint \boldsymbol{B}\cdot d\boldsymbol{A} = \frac{2\pi m}{c(a^2+z^2)^{3/2}}\int_0^{\tan^{-1}(a/z)} (2\cos\theta)((a^2+v^2)\sin\theta\, d\theta) = \frac{\pi m}{c(a^2+z^2)^{1/2}}\sin^2\tan^{-1}(a/z). \tag{2.4.4}$$

Letting $z = -vt$ and simplifying, we find

$$\oiint \boldsymbol{B}\cdot d\boldsymbol{A} = \frac{2\pi m a^2}{c(a^2+v^2t^2)^{3/2}}. \tag{2.4.5}$$

Thus,

$$\mathcal{E}(t) = -\frac{1}{c}\frac{\partial}{\partial t}\oiint \boldsymbol{B}\cdot d\boldsymbol{A} = \frac{6\pi m v^2 a^2 t}{c^2(a^2+v^2t^2)^{5/2}}. \tag{2.4.6}$$

If the loop has resistance R, then Ohm's law tells us the current will be $I = \mathcal{E}/R$, so we find

$$P(t) = \frac{1}{R}\frac{36\pi^2(mv^2)^2a^4t^2}{c^4(a^2+v^2t^2)^5}. \tag{2.4.7}$$

For the last part, the torque on the loop about its center is

$$\boldsymbol{\tau} = (a \cdot (2\pi a)\lambda \cdot \mathcal{E}(t))\,\hat{\boldsymbol{z}}. \tag{2.4.8}$$

The loop will attain its largest angular momentum when the torque vanishes, and evidently this occurs at $t = 0$. To find $L_{\max}$, we integrate $\boldsymbol{\tau}$ up to $t = 0$:

$$L_{\max} = \frac{6\pi m v^2 \lambda}{c^2 a^2}\hat{\boldsymbol{z}} \int_{-\infty}^{0} \frac{t\,dt}{\left(1 + \frac{v^2}{a^2}t^2\right)^{5/2}} = -\frac{2\pi m\lambda}{c^2}\hat{\boldsymbol{z}}. \tag{2.4.9}$$

In the previous problem, some fields moved around, and suddenly there was energy and angular momentum in a mechanical system. If we want to make sense of this and still have conservation laws, we need to determine how the electromagnetic field can carry conserved quantities.

Before we can do this, we note that Ampére's law is not complete as we have stated it. Just as a changing magnetic field contributes to the curl of an electric field, a changing electric field contributes to the curl of a magnetic field. The full Ampére law is

$$\nabla \times \boldsymbol{B} = \frac{1}{c}\left(4\pi\boldsymbol{j} + \frac{\partial \boldsymbol{E}}{\partial t}\right). \tag{2.4.10}$$

For historical reasons, the additional term is sometimes called the displacement current.

Now we return to the question of field energy. We have already worked this out in the electrostatic case, finding an energy density of $u = \frac{1}{8\pi}E^2$. We might expect a similar statement for the magnetic field, but until now we have been unable to address how the magnetic field stores energy, because the Lorentz force $q\frac{\boldsymbol{v}}{c} \times \boldsymbol{B}$ does no work. Using Faraday's law, we have

$$\frac{\partial}{\partial t}B^2 = 2\boldsymbol{B} \cdot \frac{\partial \boldsymbol{B}}{\partial t} = -2c\boldsymbol{B} \cdot (\nabla \times \boldsymbol{E}) = -2c\nabla \cdot (\boldsymbol{E} \times \boldsymbol{B}) - 2c\boldsymbol{E} \cdot (\nabla \times \boldsymbol{B}). \tag{2.4.11}$$

Using Ampére's law, we find

$$\frac{\partial}{\partial t}B^2 = -2c\nabla \cdot (\boldsymbol{E} \times \boldsymbol{B}) - 8\pi\boldsymbol{E} \cdot \boldsymbol{j} - 2\boldsymbol{E} \cdot \frac{\partial \boldsymbol{E}}{\partial t}. \tag{2.4.12}$$

Now we can examine each term on the right hand side. The final term is just $-\frac{\partial}{\partial t}E^2$, so we have in fact derived an expression for the time derivative of $E^2 + B^2$. We will divide through by 8π since we know this gives the correct energy density of the electric field. Taking the whole expression to be an energy density

$$u = \frac{1}{8\pi}\left(E^2 + B^2\right), \tag{2.4.13}$$

we have

$$\frac{\partial u}{\partial t} = -\nabla \cdot \boldsymbol{S} - \boldsymbol{E} \cdot \boldsymbol{j}. \tag{2.4.14}$$

The second term on the right is energy lost to mechanical motion of charges as the electric field accelerates them. The first term is new and mysterious; we have defined the Poynting vector $\boldsymbol{S} = \frac{c}{4\pi}(\boldsymbol{E} \times \boldsymbol{B})$. In the absence of current, $\boldsymbol{j} = 0$, we have

$$\frac{\partial u}{\partial t} + \nabla \cdot \boldsymbol{S} = 0. \tag{2.4.15}$$

This is a continuity equation for energy if we identify $\boldsymbol{S}$ as an energy current. Since $\boldsymbol{S}$ depends only on the fields, it describes how energy can be carried by electrodynamics in vacuum (or in media).

One conserved quantity down, two to go. We can consider momentum next; this is fairly easy, because momentum and energy are joined at the hip by relativity. The conservation of energy momentum $\partial_\mu P^\mu = 0$ can be expanded as

$$\frac{1}{c^2}\frac{\partial u}{\partial t} + \nabla \cdot \boldsymbol{g} = 0, \tag{2.4.16}$$

where $\boldsymbol{g}$ is the momentum density. Using (2.4.14), we find that in vacuum (where the momentum of charged particles can be ignored), we have

$$\boldsymbol{g} = \frac{1}{c^2}\boldsymbol{S}. \tag{2.4.17}$$

This is the momentum density of the electromagnetic field itself. From this we can immediately compute the angular momentum density about some origin,

$$\boldsymbol{\ell} = \frac{1}{c^2}\boldsymbol{r} \times \boldsymbol{S}. \tag{2.4.18}$$

Problem 2.17 (M12E1)
A long thin non-conducting cylinder of radius r and height $h \gg r$ is concentric with a line charge of charge per unit length $-\lambda$. The cylinder has a uniform surface charge density with equal and opposite total charge per unit length $+\lambda$. The cylinder is free to rotate about its symmetry axis and has moment of inertia per unit length I/h. At times $t < 0$ the cylinder is at rest and a spatially uniform axial external magnetic field $B_0\hat{\boldsymbol{z}}$ is present, as shown in the figure. At time $t = 0$, the externally applied field is ramped down to zero.

a) Compute the torque on the cylinder in terms of $\frac{dB_z}{dt}$, with B_z the (approximately uniform) axial magnetic field *within* the cylinder.

b) Find the angular velocity of the cylinder after the external field is reduced to zero, noting that the final field within the cylinder will be non-zero. Express your answer in terms of λ, r, B_0, I, and/or h and whatever fundamental constants are required.

c) Recalling that the density of linear momentum stored in the electromagnetic field is proportional to the Poynting vector, express the angular momentum of the initial state. Demonstate that the total angular momentum (mechanical plus electromagnetic) is conserved between the initial and final states.

The electric field induced in the cylinder is $\frac{1}{c}\frac{r}{2}\frac{dB_z}{dt}\hat{\boldsymbol{\theta}}$. This means the total torque is

$$\boldsymbol{\tau} = \frac{r^2 h\lambda}{2c}\frac{dB_z}{dt}\hat{\boldsymbol{z}}. \tag{2.4.19}$$

The rotation of the cylinder leads to a magnetic field within the cylinder. We can compute this from Ampére's law. If we draw a rectangle with one side going through the cylinder, the current piercing this rectangle will be $(\frac{\lambda h}{2\pi r})(\omega r)$, so the field generated by the cylinder is

$$\boldsymbol{B}_{\text{cyl}} = \frac{\lambda\omega}{2\pi c}\hat{\boldsymbol{z}}. \tag{2.4.20}$$

We thus need to solve

$$I\omega = \frac{r^2 h\lambda}{2c}\left(B_0 - \frac{\lambda\omega}{2\pi c}\right). \tag{2.4.21}$$

The result is

$$\omega = \frac{r^2 h\lambda B_0}{2Ic}\left(1 + \frac{\lambda^2 r^2 h}{4\pi I c^2}\right)^{-1}. \tag{2.4.22}$$

The initial electric field is

$$\boldsymbol{E}(s) = \begin{cases} -\frac{2\lambda}{s}\hat{\boldsymbol{s}} & s < r \\ 0 & s \geq r \end{cases}. \tag{2.4.23}$$

The Poynting vector in the cylinder is thus

$$\boldsymbol{S} = \frac{c}{4\pi}\boldsymbol{E}\times\boldsymbol{B} = \frac{\lambda c B_0}{2\pi s}\hat{\boldsymbol{\theta}}, \tag{2.4.24}$$

and so the angular momentum density is

$$\boldsymbol{\ell} = \frac{1}{c^2}\boldsymbol{r}\times\boldsymbol{S} = \frac{\lambda B_0}{2\pi c}\hat{\boldsymbol{z}}, \tag{2.4.25}$$

giving a total initial angular momentum of $L = \frac{\lambda B_0 r^2 h}{2c}$.

For the final electromagnetic angular momentum, we simply replace B_0 with $\frac{\lambda\omega}{2\pi c}$. The mechanical angular momentum is $I\omega$. In total,

$$L = \left(I + \frac{\lambda^2 r^2 h}{4\pi c^2}\right)\omega = \frac{r^2 h\lambda B_0}{2c}. \tag{2.4.26}$$

And would ya look at that, physics actually worked.

Problem 2.18 (J07E2)
A hollow spherical shell centered at the origin has radius a and a total electric charge $Q > 0$ uniformly distributed over its surface. The shell is slowly spun up to an angular velocity $\omega = \omega_0\hat{\boldsymbol{z}}$ (where $\omega_0 > 0$) over a period of time $\tau \gg a/c$, where c is the speed of light, so radiation effects can be ignored.

a) To linear order in $\frac{d\omega}{dt}$, find expressions for the electromagnetic fields $\boldsymbol{E}(\boldsymbol{r})$ and $\boldsymbol{B}(\boldsymbol{r})$ throughout space, as functions of ω and $\frac{d\omega}{dt}$. Make a qualitatively correct sketch showing the pattern of electric field lines in the plane $z = 0$. Indicate the direction of rotation of the charged shell on your plot.

b) After the angular velocity ω_0 is reached, what is the total angular momentum $\hat{\boldsymbol{L}}$ stored in the electromagnetic fields?

We are working to linear order in $\frac{d\omega}{dt}$, so we start by finding the fields which contain no powers of $\frac{d\omega}{dt}$, i.e., the fields of a sphere rotating at constant ω. The electric field is simply

$$\boldsymbol{E} = \begin{cases} \frac{Q}{r^2}\hat{\boldsymbol{r}} & r > a \\ 0 & r < a \end{cases}. \tag{2.4.27}$$

To find the magnetic field, we need to integrate the current

$$\boldsymbol{j} = \frac{Q}{4\pi a}\omega \sin\theta \hat{\boldsymbol{\phi}}. \tag{2.4.28}$$

over the sphere. One could probably do so directly, but there are far more fun things to do with one's day, such as elective dental surgery. Instead, we will borrow a useful solution technique from electrostatics, separation of Laplace's equation. Everywhere except the sphere we have $\nabla \times \boldsymbol{B} = 0$, so we could write $\boldsymbol{B} = -\nabla\Phi$ off the sphere. We will need to split this into an inner and an outer solution, and the current distribution on the sphere relates the two. Indeed, Ampére's law implies

$$B_{\theta,\text{out}} - B_{\theta,\text{in}} = \frac{Q\omega}{ac}\sin\theta. \tag{2.4.29}$$

Furthermore, $\nabla \cdot \boldsymbol{B} = 0$ implies that B_r is continuous across the sphere. With this information, we can write two magnetic potentials

$$\Phi_{\text{in}} = \sum_{\ell=0}^{\infty}\left(A_\ell^{\text{in}} r^\ell + \frac{B_\ell^{\text{in}}}{r^{\ell+1}}\right)P_\ell(\cos\theta) \qquad \Phi_{\text{out}} = \sum_{\ell=0}^{\infty}\left(A_\ell^{\text{out}} r^\ell + \frac{B_\ell^{\text{out}}}{r^{\ell+1}}\right)P_\ell(\cos\theta). \tag{2.4.30}$$

Boundary conditions at 0 and infinity require $B_\ell^{\text{in}} = A_\ell^{\text{out}} = 0$. Continuity of B_r implies

$$B_\ell^{\text{out}} = -\frac{\ell}{\ell+1}a^{2\ell+1}A_\ell^{\text{in}}, \tag{2.4.31}$$

and the current density implies

$$B_\ell^{\text{out}} P_\ell'(\cos\theta)\sin\theta - A_\ell^{\text{in}} a^{2\ell+1} P_\ell'(\cos\theta)\sin\theta = \frac{Q\omega}{c}a^{\ell+1}\sin\theta. \tag{2.4.32}$$

Clearly we should only use the $\ell = 1$ terms, for which $P_\ell'(\cos\theta) = 1$. Then we have

$$B_1^{\text{out}} = -\frac{1}{2}a^3 A_\ell^{\text{in}}, \qquad B_\ell^{\text{out}} - A_\ell^{\text{in}} a^3 = \frac{Q\omega}{c}a^2. \tag{2.4.33}$$

We easily solve this system and find

$$\Phi_{\text{in}} = -\frac{2Q\omega}{3c}\frac{r}{a}\cos\theta, \qquad \Phi_{\text{out}} = \frac{Q\omega}{3c}\frac{a^2}{r^2}\cos\theta. \tag{2.4.34}$$

This gives a magnetic field

$$\boldsymbol{B} = \begin{cases} \frac{2Q\omega}{3ca}(\cos\theta\hat{\boldsymbol{r}} - \sin\theta\hat{\boldsymbol{\theta}}) & r < a \\ \frac{Q\omega a^2}{3cr^3}(2\cos\theta\hat{\boldsymbol{r}} + \sin\theta\hat{\boldsymbol{\theta}}) & r > a \end{cases}. \tag{2.4.35}$$

Finally, we find the electric field induced by the change in this magnetic field. If only we knew the vector potential, we could immediately write

$$\boldsymbol{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t}, \tag{2.4.36}$$

and the second piece would be the part linear in $\frac{d\omega}{dt}$. It is not too difficult to determine a vector potential from the magnetic field. If only $A_\phi \neq 0$, then we would have

$$\nabla \times \boldsymbol{A} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\phi)\,\hat{\boldsymbol{r}} - \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)\hat{\boldsymbol{\theta}}. \tag{2.4.37}$$

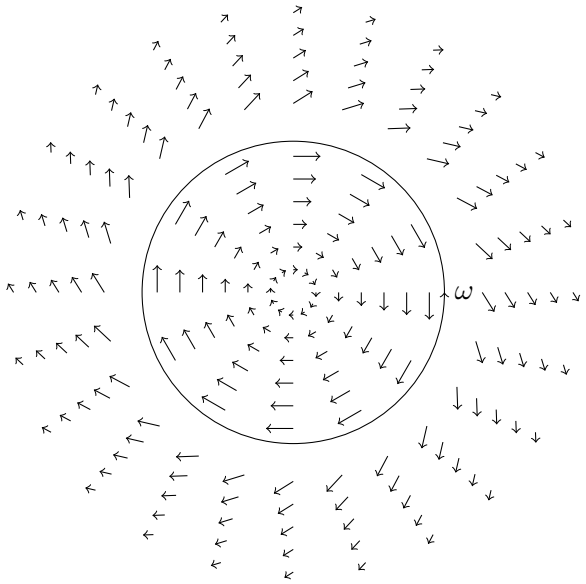
Comparing this with $\boldsymbol{B}$, we see that

$$A_\phi = \begin{cases} \frac{Q\omega}{3ca} r\sin\theta & r < a \\ \frac{Q\omega a^2}{3c}\frac{\sin\theta}{r^2} & r > a \end{cases}. \tag{2.4.38}$$

It immediately follows that

$$\boldsymbol{E} = \begin{cases} -\frac{Q}{3ca}\frac{d\omega}{dt} r\sin\theta\hat{\boldsymbol{\phi}} & r < a \\ \frac{Q}{r^2}\hat{\boldsymbol{r}} - \frac{Qa^2}{3c}\frac{d\omega}{dt}\frac{\sin\theta}{r^2} & r > a \end{cases}. \tag{2.4.39}$$

In the plane $z = 0$ we have $\theta = \frac{\pi}{2}$, so these expressions simplify somewhat. The $\boldsymbol{E}$ field in this plane is shown below.



After the shell stops rotating, the $\frac{d\omega}{dt}$ correction to the electric field vanishes, and so

$$\boldsymbol{g} = \frac{1}{4\pi c}\boldsymbol{E}\times\boldsymbol{B} = \begin{cases} 0 & r < a \\ \frac{Q^5\omega_0 a^2}{12\pi c^2 r^5}\sin\theta\hat{\boldsymbol{\phi}} & r > a \end{cases}. \tag{2.4.40}$$

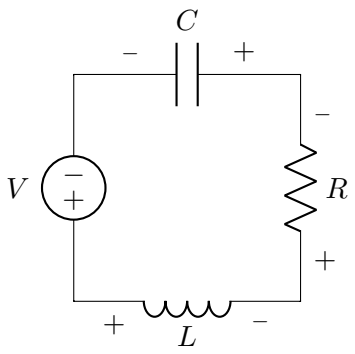


Figure 2.8: A driven RLC circuit.

This gives an angular momentum density

$$\boldsymbol{\ell} = \boldsymbol{r} \times \boldsymbol{g} = \begin{cases} 0 & r < a \\ \frac{Q^2\omega_0 a^2}{12\pi c^2 r^4} \sin\theta(\sin\theta\hat{\boldsymbol{z}} - \cos\theta\cos\phi\hat{\boldsymbol{x}} - \cos\theta\sin\phi\hat{\boldsymbol{y}}) & r > a \end{cases}, \tag{2.4.41}$$

where we have expanded $\hat{\boldsymbol{\theta}}$ in terms of Cartesian basis vectors. Clearly only the $\hat{\boldsymbol{z}}$ component will be nonvanishing in the integral, and we find

$$\boldsymbol{L} = \frac{Q^2\omega_0 a^2}{6c^2}\hat{\boldsymbol{z}} \int_a^\infty r^2\,dr \int_0^\theta (\sin\theta\,d\theta)\frac{\sin^2\theta}{r^4} = \frac{Q^2\omega_0 a}{9c^2}\hat{\boldsymbol{z}}. \tag{2.4.42}$$

Electromagnetic induction opens up the possibility of interesting circuits. We have already derived Ohm's law for resistors,

$$V = IR = R\frac{dQ}{dt}. \tag{2.4.43}$$

Capacitors have a voltage related to charge itself, rather than its derivative:

$$V = \frac{Q}{C}, \tag{2.4.44}$$

where C is capacitance. Using a coil of wire one can form an inductor, which responds to changes in current via

$$V = L\frac{d^2Q}{dt^2}, \tag{2.4.45}$$

where L is inductance.

Figure 2.8 depicts a circuit containing all of these elements, in addition to a battery with voltage V. Since the total change in voltage around the circuit must vanish, we find

$$V = \frac{Q}{C} + R\frac{dQ}{dt} + L\frac{d^2Q}{dt^2}. \tag{2.4.46}$$

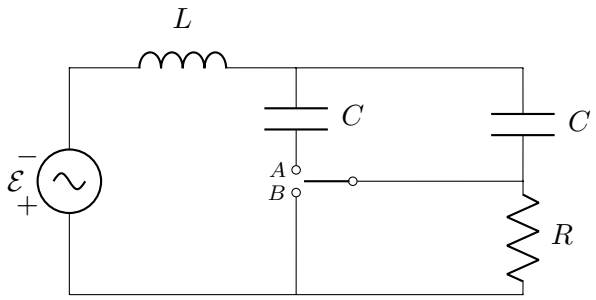
This equation is easy to solve by standard methods. It takes the form of a damped oscillator with undamped frequency $\frac{1}{\sqrt{LC}}$.

Now imagine replacing the DC battery with an AC power source, which provides a voltage $V = V_0 e^{i\omega t}$. We can use the ansatz $Q = \frac{I_0}{i\omega} e^{i\omega t}$, and find

$$V = \frac{1}{i\omega C} I_0 + R I_0 + i\omega L I_0 = \left(\frac{1}{i\omega C} + R + i\omega L \right) I_0. \tag{2.4.47}$$

We call the quantity in parentheses the impedance of the circuit. The real part is resistance, and the imaginary part is called reactance and denoted X. Note that if the driving frequency ω approaches one of the characteristic frequencies of the circuit, the impedance tends to zero; in this case we say the circuit is resonating.

Problem 2.19 (J13E3)
Complex Impedances



a) Consider the circuit above. The switch can be set in any of three positions, A, B, or open (unconnected). The source supplies a voltage $\mathcal{E}(t) = \mathcal{E}_0 e^{i\omega t}$.

When the switch is connected to A, find the frequency ω that maximizes the current through the resistor R.

b) If we then flip the switch to the B position, what is the average power dissipation in the circuit (ignoring transient effects)?

c) We now open the switch to the middle position. Find the value of the resistor R that will drop the amplitude to 1/2 the value you found in part a), at the same frequency ω that you found in part a).

d) Suppose that the inductor, of inductance L, is constructed from a solenoid with N turns over a length ℓ, whose axis of symmetry lies on the $\hat{x}$ axis.

Express the cross sectional area of the solenoid in terms of the inductance L, the number of turns N, the length ℓ and any fundamental constants.

When the switch is connected to A, the two capacitors run in parallel, giving an equivalent capacitance $2C$. The impedance of the circuit is then

$$Z = R + i\omega L + \frac{1}{2i\omega C}. \tag{2.4.48}$$

We need to find the value of ω which minimizes the magnitude of the impedance. It is easy to see that this is $\omega = \frac{1}{\sqrt{2LC}}$. This makes the reactance vanish, so we have an current with amplitude

$$I = \frac{\mathcal{E}}{R}. \tag{2.4.49}$$

After flipping the switch to B, the resistor is shorted out and we have an LC circuit, so no power is dissipated.

With the switch in the middle position, the impedance is

$$Z = R + i\omega L + \frac{1}{i\omega C} = R - i\sqrt{\frac{L}{2C}}. \tag{2.4.50}$$

We thus need

$$\sqrt{R^2 + \frac{L}{2C}} = 2R, \tag{2.4.51}$$

which implies $R = \sqrt{\frac{L}{6C}}$.

If we run a current I through a solenoid with N turns over a length ℓ, the magnetic field within the solenoid will be

$$\boldsymbol{B} = \frac{4\pi N I}{\ell c}\hat{\boldsymbol{x}}. \tag{2.4.52}$$

From Faraday's law, we find

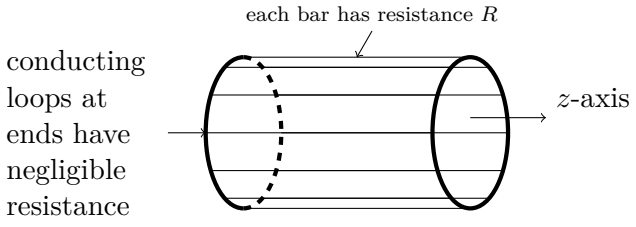
$$L = \frac{4\pi N^2}{\ell c^2}A, \tag{2.4.53}$$

where A is the cross-sectional area. Thus, $A = \frac{L\ell c^2}{4\pi N^2}$.

Problem 2.20 (M10E3)
An induction motor consists of two elements. The "stator" produces a time dependent magnetic field with a direction that rotates with angular frequency ω_s, determined by the frequency of the AC current source. Take it to be

$$(B_x(t), B_y(t), B_z(t)) = B_0(\cos\omega_s t, \sin\omega_s t, 0). \tag{2.4.54}$$

The second component is the rotor, which in one design resembles a cylindrical "cage" that is free to rotate about its axis (the z-axis), formed by $N \gg 1$ equally spaced conducting bars of length ℓ, each with resistance R, connected by a metal ring of radius r at each end of the cylinder (which has negligable resistance).



Electrical currents in the cage may be described as N independent current loops, each one defined by two adjacent bars, connected by the rings at each end of the cage. Assume the self-inductance and mutual inductance of these loops are negligible (much less than R/ω_s). The positions of the bars are defined by the line segments

$$(x, y, z) = (r\cos\theta_j, r\sin\theta_j, z), \qquad 0 < z < L, \qquad j = 1, 2, \ldots, N, \tag{2.4.55}$$

with $\theta_j = 2\pi j/N + \omega_r t$ (the rotor rotates with angular velocity $\omega_r \neq \omega_s$).

a) As a function of time, what is the induced emf across bar j?

b) Find the time-averaged torque exerted on the rotor as a function of ω_r.

The flux through bar j is

$$\Phi_j = B_0\cos((\omega_s + \omega_r)t + 2\pi j/N)L\frac{2\pi r}{N}, \tag{2.4.56}$$

so the induced emf is

$$\mathcal{E}_j = \frac{2\pi r L B_0(\omega_s + \omega_r)}{Nc}\sin((\omega_s + \omega_r)t + 2\pi j/N). \tag{2.4.57}$$

The current through the jth bar is

$$I_j = \frac{\mathcal{E}_j - \mathcal{E}_{j-1}}{R} = \frac{4\pi^2 r L B_0(\omega_s + \omega_r)}{N^2 Rc}\cos((\omega_s + \omega_r)t + 2\pi j/N), \tag{2.4.58}$$

and the force on this current is

$$\boldsymbol{F}_j = \frac{1}{c}I_j L\hat{\boldsymbol{z}}\times\boldsymbol{B} = \frac{4\pi^2 r L^2 B_0(\omega_s + \omega_r)}{N^2 Rc^2}\cos((\omega_s+\omega_r)t+2\pi j/N)\times(-\sin\omega_s t, \cos\omega_s t, 0). \tag{2.4.59}$$

The torque on the jth bar will be

$$\boldsymbol{\tau}_j = \frac{4\pi^2 r^2 L^2 B_0(\omega_s + \omega_r)}{N^2 Rc^2}\cos^2((\omega_s + \omega_r)t + 2\pi j/N)\hat{\boldsymbol{z}}, \tag{2.4.60}$$

and so the time-averaged total torque is

$$\langle\boldsymbol{\tau}\rangle = \frac{4\pi^2 r^2 L^2 B_0(\omega_s + \omega_r)}{2NRc^2}. \tag{2.4.61}$$

2.5 Electromagnetic Waves

Changing magnetic fields induce electric fields. Changing electric fields induce magnetic fields. This could get real complicated, real fast.

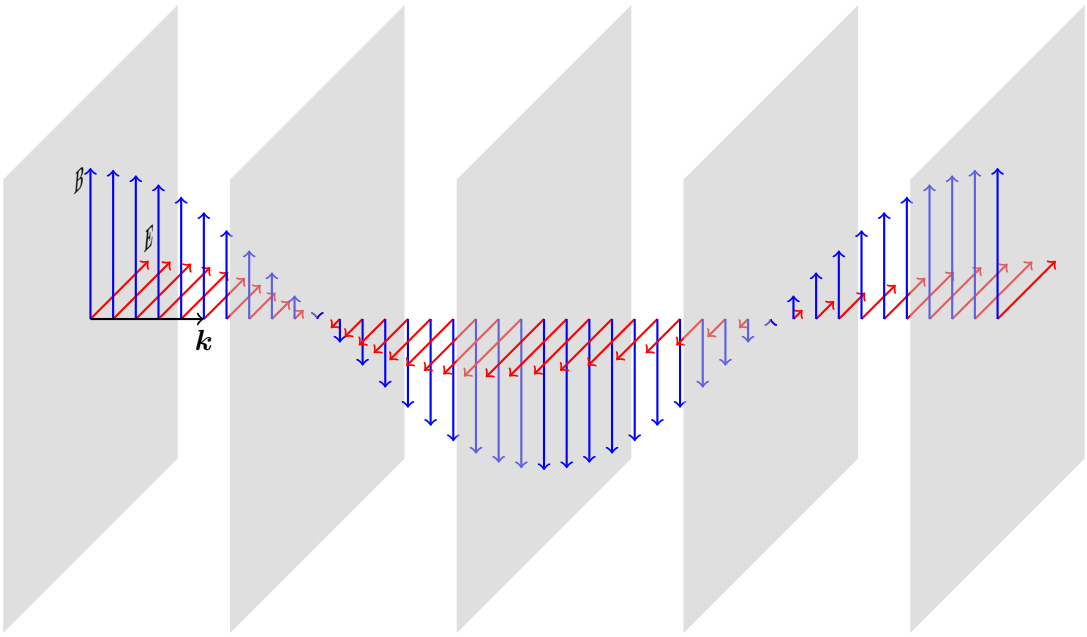


Figure 2.9: An electromagnetic plane wave.

And it does, but that's okay. Taking the curl of Faraday's equation in vacuum, we find

$$\frac{1}{c^2}\frac{\partial^2 \boldsymbol{E}}{\partial^2 t} = \nabla^2 \boldsymbol{E}. \tag{2.5.1}$$

Likewise, taking the curl of Ampére's equation in vacuum, we find

$$\frac{1}{c^2}\frac{\partial^2 \boldsymbol{B}}{\partial^2 t} = \nabla^2 \boldsymbol{B}. \tag{2.5.2}$$

This is the wave equation for a wave that propagates at speed c. *Fiat lux* and all that crap.

The ansatz $\boldsymbol{E} = \boldsymbol{E}_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}$ solves the equation whenever $\omega^2 = c^2k^2$; to satisfy Gauss's law, we additionally need $\boldsymbol{k}\cdot\boldsymbol{E}_0 = 0$. This is called a plane wave. The corresponding solution for $\boldsymbol{B}$ is fixed by Maxwell's equations to be $\boldsymbol{B} = \boldsymbol{B}_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}$, where $\boldsymbol{B}_0 = \frac{c\boldsymbol{k}\times\boldsymbol{E}_0}{\omega} = \hat{\boldsymbol{k}}\times\boldsymbol{E}_0$. Note that the electric and magnetic fields in a plane wave have the same magnitude – after all, why wouldn't they? It would be pretty silly to use units where this isn't the case.

The wavevector $\boldsymbol{k}$ specifies the direction of propagation, and the fields are both perpendicular to $\boldsymbol{k}$. Both fields oscillate in phase with one another, in time and in space. The geometry is depicted in Figure 2.9.

Problem 2.21 (M07E3)
A plane electromagnetic wave with electric field E_0 and frequency ω is incident at normal incidence on a metal film with conductivity σ, $\epsilon = 1$ and $\mu = 1$.

a) Calculate the electric and magnetic fields as a function of distance x into the conductor.

b) Show that the energy lost by the electromagnetic wave in a small distance Δx inside the conductor is equal to the ohmic heat deposited by the electromagnetic wave in that distance.

We derived the wave equation for the electromagnetic field in a vacuum, but now we are in a conductor with $\boldsymbol{j} = \sigma \boldsymbol{E}$. Carrying through the derivation as before, we find

$$\begin{aligned} \nabla^2 \boldsymbol{E} &= \frac{1}{c^2}\frac{\partial^2 \boldsymbol{E}}{\partial^2 t} + \frac{4\pi\sigma}{c^2}\frac{\partial \boldsymbol{E}}{\partial t}, \\ \nabla^2 \boldsymbol{B} &= \frac{1}{c^2}\frac{\partial^2 \boldsymbol{B}}{\partial^2 t} + \frac{4\pi\sigma}{c^2}\frac{\partial \boldsymbol{B}}{\partial t}. \end{aligned} \tag{2.5.3}$$

Let the surface of the conductor be the yz plane, and let $\boldsymbol{k} = k\hat{\boldsymbol{x}}$. The fields on the surface of the conductor from the incident plane wave will be $\boldsymbol{E} = (E_0\hat{\boldsymbol{y}})e^{-i\omega t}$ and $\boldsymbol{B} = (E_0\hat{\boldsymbol{z}})e^{-i\omega t}$. Clearly continuity requires that the wave which propagates into the conductor also has frequency ω, but its wavevector can be $\boldsymbol{k'} = k'\hat{\boldsymbol{x}}$. Substituting $(E_0\hat{\boldsymbol{x}})e^{i(\boldsymbol{k'}\cdot\boldsymbol{x}-\omega t)}$, we have

$$k' = \frac{1}{c}\sqrt{\omega^2 + 4\pi i\sigma\omega}. \tag{2.5.4}$$

Thus, if we move a distance x into the conductor, the fields are

$$\begin{aligned} \boldsymbol{E} &= (E_0\hat{\boldsymbol{y}})e^{i(k'x-\omega t)}, \\ \boldsymbol{B} &= (E_0\hat{\boldsymbol{z}})e^{i(k'x-\omega t)}. \end{aligned} \tag{2.5.5}$$

Since k' is complex, there will be dissipation as well as oscillation as the wave penetrates.

Letting $\operatorname{Im} k' = \kappa$, the energy density of the wave is

$$u = \frac{1}{8\pi}(E^2 + B^2) = \frac{E_0^2}{4\pi}e^{-2\kappa x}. \tag{2.5.6}$$

Thus, the energy lost per unit volume is approximately $\frac{E_0^2}{2\pi}\kappa\Delta x$ for small Δx. Assuming $r\pi\sigma \ll \omega$, we have $\kappa \approx \frac{2\pi\sigma}{c}$, so this is

$$\Delta u \approx -\frac{\sigma}{c}E_0^2\Delta x. \tag{2.5.7}$$

The power dissipated in a cube of conductor with side length d is

$$P = VI \approx (E_0 d)(\sigma E_0 d^3) = \sigma E_0 d^3, \tag{2.5.8}$$

so the power dissipated per volume is σE_0 near the surface. The time the wave has spent in the conductor is $\frac{\Delta x}{c}$, so the total energy dissipated as ohmic heat indeed balances the energy lost to current.

Problem 2.22 (J99E1)
It is well known that a charged particle cannot be held at rest by purely electrostatic fields. In your answers below, you will give a (simple) classical explanation of how a neutral atom of polarizability α can be "trapped" at the focus of a laser beam.

a) First, ignore magnetic interactions, and deduce that there is a (time-averaged) trapping force dependent on the electric field of the laser.

b) Atoms have some probability of absorbing photons from the laser beams, thereby being kicked along the direction of the beam. This processes can be modelled classically by supposing that the polarizability of the atom has an imaginary part: $\alpha = \alpha' + i\alpha''$. Deduce the force on an atom along the direction of propagation of a linearly polarized plane electromagnetic wave in terms of α'', the imaginary (absorptive) part of the polarizability.

c) For an idealized atom with a single natural frequency ω_0, deduce the ratio α'/α'' at the frequency ω for which the real part, α', of the polarizability is a maximum. For this, you may use a classical model of an atom as an electron on a spring of frequency ω_0, subject to a damping force $-\gamma m\dot{x}$, where $\gamma \ll \omega_0$ is the reciprocal of the lifetime of the 'excited state'.

In practice, the trapping force a) must be larger than the longitudinal force b). This requires the laser beam to be tightly focused.

Let the electric field of the laser be $\boldsymbol{E} = \boldsymbol{E}_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}$, and so the atom will have a dipole moment $\boldsymbol{p} = \alpha \boldsymbol{E}_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}$. The potential energy of the atom in the field is given by

$$U = -\boldsymbol{p}\cdot\boldsymbol{E} = -\alpha E_0^2 \cos^2(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t), \tag{2.5.9}$$

where we have used the real parts of $\boldsymbol{p}$ and $\boldsymbol{E}$. Time-averaging, we find $\langle U\rangle = -\frac{1}{2}\alpha E_0^2$. For a plane wave this would be a constant, but we are actually working with a highly focused laser beam. We should replace E_0^2 by the intensity I of the beam, which has a maximum at the position of the atom. Then indeed there is a time-averaged trapping force

$$\langle \boldsymbol{F}\rangle = -\frac{1}{2}\alpha \nabla I. \tag{2.5.10}$$

Now let the polarizability be $\alpha = \tilde{\alpha} e^{i\theta}$. Then we have

$$U = -\boldsymbol{p}\cdot\boldsymbol{E} = -\tilde{\alpha} E_0^2 \cos(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)\cos(\theta + \boldsymbol{k}\cdot\boldsymbol{x} - \omega t) \tag{2.5.11}$$

$$= -\alpha' E_0^2 \cos^2(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t) + \alpha'' E_0^2 \sin(2(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)). \tag{2.5.12}$$

The second term leads to a longitudinal component in the force,

$$\boldsymbol{F} = -2\alpha'' E_0^2 \boldsymbol{k} \cos(2(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)). \tag{2.5.13}$$

This vanishes in the time average, but if it is too large it could knock the atom out of the potential well.

If we treat an atom as an electron behaving according to

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = -\frac{E_0}{m} e^{i\omega t}, \tag{2.5.14}$$

we can use the ansatz $x = x_0 e^{i\omega t}$ and solve, obtaining

$$x(t) = -\frac{E_0}{m}\frac{1}{\omega_0^2 - \omega^2 - i\gamma\omega} e^{i\omega t}. \tag{2.5.15}$$

The polarizability is then

$$\alpha = \frac{1}{m}\frac{1}{\omega_0^2 - \omega^2 - i\gamma\omega}. \tag{2.5.16}$$

Maximizing the real part, we find $\omega = \sqrt{\omega_0^2 - \gamma\omega_0}$, and substituting this in we find

$$\alpha \propto \frac{1}{\gamma\omega_0 - i\gamma\sqrt{\omega_0^2 - \gamma\omega_0}}, \tag{2.5.17}$$

and it follows that

$$\alpha'/\alpha'' = \left(1 - \frac{\gamma}{\omega_0}\right)^{-1/2}. \tag{2.5.18}$$

Since $\gamma \ll \omega_0$, we see that the real and imaginary parts of the polarizability are nearly equal in magnitude.

Problem 2.23 (J99E2)
When a charged particle (of mass m and charge e) interacts with a linearly polarized plane wave (with electric field $E_x = E_0\cos(kz - \omega t)$), the particle's motion includes a transverse oscillation. In the frame in which the particle is at rest on average, the motion is purely transverse if $eE_0/m\omega c \ll 1$, as may be assumed. Hence, the particle has transverse momentum, while the wave carries only longitudinal momentum. How is Newton's 3rd law satisfied in this situation?

The friends are invisible and all around us; momentum will be conserved if we properly account for the momentum due to the field of the particle. One might be tempted to work out the magnetic field of the particle, but the argument needs to work for a particle of arbitrarily large mass. If we take $m \to \infty$, the particle will have negligible velocity and won't generate a $\boldsymbol{B}$ field. The extra momentum should instead come from the interaction of the particle's $\boldsymbol{E}$ field with the $\boldsymbol{B}$ field of the wave, given by

$$\boldsymbol{g}_{\text{extra}} = \frac{1}{4\pi c}\left(\frac{e}{r^2}\hat{\boldsymbol{r}}\right) \times (E_0\cos(kz - \omega t)\hat{\boldsymbol{y}}) = \frac{eE_0}{4\pi c}\cos(kz - \omega t)\frac{x\hat{\boldsymbol{z}} - z\hat{\boldsymbol{x}}}{(x^2 + y^2 + z^2)^{3/2}}. \tag{2.5.19}$$

Upon integration, most terms vanish; the only one which can survive is the one proportional to $z\sin(kz)\sin(\omega t)$. We find

$$P_{x,\text{extra}} = -\frac{eE_0}{4\pi c}\sin(\omega t)\int d^3\boldsymbol{r}\frac{z\sin(kz)}{r^3}. \tag{2.5.20}$$

We can perform the integral in spherical coordinates, giving

$$\int d^3\boldsymbol{r}\frac{z\sin(kz)}{r^3} = 2\pi\int_0^\infty r^2\,dr\int_{-1}^1 d(\cos\theta)\frac{\cos\theta\sin(kr\cos\theta)}{r^2} \tag{2.5.21}$$

$$= 2\pi\int_0^\infty dr\frac{2\sin(kr) - 2kr\cos(kr)}{k^2r^2} \tag{2.5.22}$$

$$= 2\pi\left(-\frac{2\sin(kr)}{k^2r}\right)_0^\infty \tag{2.5.23}$$

$$= \frac{4\pi}{k}, \tag{2.5.24}$$

which gives $P_{x,\text{extra}} = -\frac{eE_0}{\omega}\sin(\omega t)$. This exactly cancels the transverse mechanical momentum.

In addition to the frequency and intensity, electromagnetic waves have a degree of freedom in their polarization. We have thus far taken the electric field to point along the $\hat{\boldsymbol{x}}$ axis, but it can point anywhere in the plane perpendicular to $\boldsymbol{k}$. The magnetic field will always be in the direction $\hat{\boldsymbol{k}} \times \boldsymbol{E}$, so we only need to specify the direction of the electric field to fully specify the polarization.

In addition to pure vertical or horizontal polarization, we could have vertical and horizontal waves superimposed with a phase shift between them, so that the electric field is

$$\boldsymbol{E} = E_0\hat{\boldsymbol{x}}\cos(kz - \omega t) + E_1\hat{\boldsymbol{y}}\cos(kz - \omega t + \phi). \tag{2.5.25}$$

If we think of this as the real part of a complex function as usual, then we find

$$\boldsymbol{E} = (E_0\hat{\boldsymbol{x}} + e^{i\phi}E_1\hat{\boldsymbol{y}})e^{i(kz-\omega t)}. \tag{2.5.26}$$

We can thus think about one of these phase-shifted superpositions as having a complex polarization vector. Generically these states are elliptically polarized, meaning the electric field vector will trace out an ellipse at any given point. If $\phi = n\pi$ the wave is linearly polarized; if $\phi = \left(n + \frac{1}{2}\pi\right)$, the wave is circularly polarized.

Problem 2.24 (M03E3)
The $\boldsymbol{E}$-vector of a plane electromagnetic wave propagating along the z-axis and having a polarization vector $\boldsymbol{e} = (\alpha_x, \alpha_y, 0)$ can be written

$$\boldsymbol{E}(\boldsymbol{r}, t) = E(\alpha_x\hat{\boldsymbol{x}} + \alpha_y\hat{\boldsymbol{y}})e^{i(kz-\omega t)} \tag{2.5.27}$$

(the polarization vector is taken to be of unit length, $\boldsymbol{\alpha} \cdot \boldsymbol{\alpha}^* = 1$). In free space, the dispersion relation is $\omega = ck$ and the wave propagates with both phase and group velocity equal to c.

Now let the wave propagate through a dilute plasma containing a density N of free mobile electrons of mass m and charge e (along with a background of compensating positive charge taken to be so massive as to be fixed in place). By solving for the motion of an electron in the electric field of the propagating wave (i.e. ignoring the effect of the wave's $\boldsymbol{B}$-field on the motion) one can infer a polarization density $\boldsymbol{P} \propto \boldsymbol{E}$. This in turn allows us to infer a frequency-dependent dielectric constant via

$$\boldsymbol{D} = \boldsymbol{E} + 4\pi\boldsymbol{P} = \epsilon(\omega)\boldsymbol{E}. \tag{2.5.28}$$

The index of refraction of plasma is then given by $n(\omega) = \sqrt{\epsilon}$.

a) Use this line of argument to compute the frequency dependence of the index of refraction $n(\omega)$ of a plasma. Turn your result into a dispersion relation $\omega(k)$ and find the limiting frequency ω_p as the wavelength of the wave goes to infinity. This cutoff frequency, below which waves cannot propagate, is called the plasma frequency.

b) Now let the plasma be subject to a static magnetic field $B_0\hat{\boldsymbol{z}}$ in the direction of propagation of the wave. Also assume that the electrons have some kinetic energy so that, in the absence of any other perturbation, they execute circular motion about the static magnetic field lines at the Larmor frequency $\omega_L = |eB_0|/m$. Extend your calculation of the dielectric constant of the

plasma to this new case. The response is different for different states of circular polarization so, for definiteness, analyze the case of right circular polarization of the propagating wave ($\boldsymbol{\alpha} = (\hat{\boldsymbol{x}} + i\hat{\boldsymbol{y}})/\sqrt{2}$).

c) Find the lowest frequency at which such a wave can propagate. Express your answer in terms of ω_p and ω_L.

The magnitude of each electron's oscillation is $\frac{eE}{m\omega^2}$. Since the position will be opposite the acceleration, the electric dipoles actually enhance the electric field, meaning we have a negative electric susceptibility,

$$\boldsymbol{P} = -\frac{Ne^2}{m\omega^2}\boldsymbol{E}. \tag{2.5.29}$$

The dielectric constant is $\epsilon(\omega) = 1 - \frac{4\pi Ne^2}{m\omega^2}$. Therefore,

$$\omega = \frac{c}{n(\omega)}k \approx c\left(1 - \frac{4\pi Ne^2}{m\omega^2}\right)^{-1/2} k. \tag{2.5.30}$$

Solving for ω, we have

$$\omega^2 = c^2k^2 + \frac{4\pi Ne^2}{m}. \tag{2.5.31}$$

The limiting frequency as $k \to \infty$ is $\omega_p^2 = \frac{4\pi Ne^2}{m}$.

We could think of the previous part as a driven free particle,

$$\ddot{x} = -\frac{eE}{m}\sin(\omega t), \tag{2.5.32}$$

whereas with the Larmor oscillations in circular polarization we have

$$\ddot{z} + \omega_L^2 z = -\frac{eE}{m}\frac{1+i}{\sqrt{2}}e^{i\omega t}, \tag{2.5.33}$$

where $z = x + iy$. This is solved by

$$z = \frac{eE}{m(\omega^2 - \omega_L^2)}\frac{1+i}{\sqrt{2}}e^{i\omega t}. \tag{2.5.34}$$

The dielectric constant is then

$$\epsilon = 1 - \frac{4\pi Ne^2}{m(\omega^2 - \omega_L^2)} = 1 - \frac{\omega_p^2}{\omega^2 - \omega_L^2}. \tag{2.5.35}$$

The lowest possible frequency occurs when $\epsilon = 0$, which corresponds to $\omega = \sqrt{\omega_L^2 + \omega_p^2}$.

Problem 2.25 (J15E2)
A Fresnel rhomb is an optical device used to convert linearly polarized light into circularly polarized light. As shown in Fig. 2.10, light hits the surface of the rhomb at normal incidence, it then undergoes two total internal reflections inside the rhomb, and then leaves the rhomb again normally.

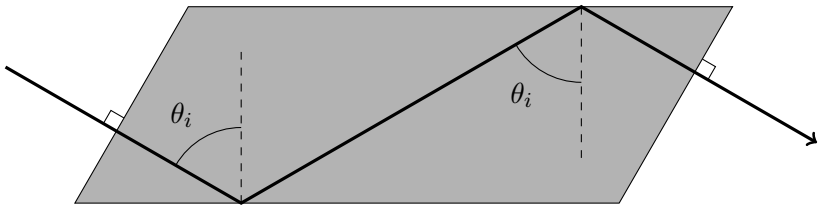


Figure 2.10: A Fresnel rhomb.

The total internal reflections are such that each reflection generates a phase difference of 45° between the component of the light-wave that is parallel and the component that is perpendicular to the plane of incidence (the plane of the page in Fig. 2.10), and so after two internal reflections a lightwave that was originally linearly polarized at 45° with respect to the plane of incidence becomes circularly polarized.

a) For a single internal reflection, find the phase shift that the reflected wave acquires relative to the incident wave assuming the electromagnetic wave is polarized in the plane of incidence.

b) Calculate the phase shift that the reflected wave acquires relative to the incident wave when the electromagnetic wave is polarized perpendicular to the plane of incidence.

c) If each of the two total internal reflections in a Fresnel rhomb occurs at an angle of incidence of $\theta_i = 53.3°$, calculate the index of refraction n of the Fresnel rhomb relative to that of the surrounding medium.

This is annoying any way you do it. There are Fresnel formulas that give the phase shifts for any kind of reflection/transmission situation, but I'd rather memorize Hamlet. You might think this case is significantly easier because there is only a reflected wave, but actually there will be an evanescent wave which is strongly damped but nonetheless penetrates the boundary. So in order to work out the boundary conditions, we have to pretend there is a transmitted component.

Since we are in a dielectric medium, the Maxwell equations become

$$\begin{aligned} \nabla \cdot \boldsymbol{D} = 0, \quad \nabla \times \boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t}, \\ \nabla \cdot \boldsymbol{B} = 0, \quad \nabla \times \boldsymbol{H} = \frac{1}{c}\frac{\partial \boldsymbol{D}}{\partial t}. \end{aligned} \tag{2.5.36}$$

Using $\boldsymbol{D} = \epsilon \boldsymbol{E}$ and $\boldsymbol{H} = \mu^{-1}\boldsymbol{B}$, we find the wave equation with phase velocity $\frac{c}{\sqrt{\epsilon\mu}} \equiv \frac{c}{n}$, and so $|\boldsymbol{B}| = n|\boldsymbol{E}|$.

We can use continuity of tangential components to fields to fix the phase shift. If we draw small rectangles around the boundary and integrate Faraday's and Ampére's laws, we find $\boldsymbol{n} \times \Delta \boldsymbol{E} = \boldsymbol{n} \times \Delta \boldsymbol{H} = 0$. This means that only the normal components of $\boldsymbol{E}$ and $\boldsymbol{H}$ can change across the boundary.

When the electric field is polarized in the plane of incidence, continuity of the tangential electric field at the reflection point requires

$$E_i e^{i\omega t} \cos\theta_i - E_r e^{i(\omega t + \phi_r)} \cos\theta_i = E_t e^{i(\omega_t + \phi_t)} \cos\theta_t, \tag{2.5.37}$$

where we have taken the normal components of E_i and E_r to both point into the medium. Similarly, continuity of the tangential component of $\boldsymbol{H}$ requires

$$\frac{n}{\mu}\left(E_i e^{i\omega t} + E_r e^{i(\omega_t + \phi_r)}\right) = E_t e^{i(\omega_t + \phi_t)}. \tag{2.5.38}$$

Eliminating the transmitted field, we find

$$\frac{E_r}{E_i} e^{i\phi_r} = \frac{\cos\theta_i - \frac{n}{\mu}\cos\theta_t}{\cos\theta_i + \frac{n}{\mu}\cos\theta_t}. \tag{2.5.39}$$

Now we have to determine $\cos\theta_t$. For this we can use Snell's law, $n \sin\theta_i = \sin\theta_t$. This implies

$$\cos\theta_t = \sqrt{1 - n^2 \sin^2\theta_i} = i\sqrt{n^2 \sin^2\theta_i - 1}. \tag{2.5.40}$$

Since we have total internal reflection, $n \sin\theta_i > 1$ and so $\cos\theta_t$ is pure imaginary. It follows that $\frac{E_r}{E_i} e^{i\phi_r} = \frac{a - ib}{a + ib}$, so indeed $E_r = E_i$, and

$$\phi_r = -2\tan^{-1}\frac{b}{a} = -2\tan^{-1}\frac{n\sqrt{n^2 \sin^2\theta_i - 1}}{\mu \cos\theta_i}. \tag{2.5.41}$$

Now we take the electric field to be perpendicular to the plane of incidence, so we move some cosines around. The boundary conditions are now

$$E_i e^{i\omega t} + E_r e^{i(\omega t + \phi_r)} = E_t e^{i(\omega_t + \phi_t)}, \tag{2.5.42}$$

$$\frac{n}{\mu}\left(E_i e^{i\omega t}\cos\theta_i - E_r e^{i(\omega_t + \phi_r)}\cos\theta_i\right) = E_t e^{i(\omega_t + \phi_t)}\cos\theta_t. \tag{2.5.43}$$

Again eliminating E_t, we find

$$\frac{E_r}{E_i} e^{i\phi_r} = \frac{\cos\theta_t - \frac{n}{\mu}\cos\theta_i}{\cos\theta_t + \frac{n}{\mu}\cos\theta_i}. \tag{2.5.44}$$

Again $\cos\theta_t$ is pure imaginary, and so

$$\phi_r = 2\tan^{-1}\frac{n\cos\theta_i}{\mu\sqrt{n^2 \sin^2\theta_i - 1}}. \tag{2.5.45}$$

In order to convert linearly polarized light into circularly polarized light, the parallel and perpendicular phases should have a difference of $n\pi + \frac{\pi}{4}$ (so that after two reflections the phase

difference is $\frac{\pi}{2}$). Note that at $\theta_i = \frac{\pi}{2}$, we have a phase difference of π, and this increases as we decrease θ_i, so in our case we want a phase difference of $\frac{5\pi}{4}$. Using this we can write

$$\tan\frac{5\pi}{8} = \tan\frac{\Delta\phi}{2} = \left(\frac{n\cos\theta_i}{\mu\sqrt{n^2\sin^2\theta_i - 1}} + \frac{n\sqrt{n^2\sin^2\theta_i - 1}}{\mu\cos\theta_i}\right)\left(1 - \frac{n^2}{\mu^2}\right)^{-1} \tag{2.5.46}$$

$$= \frac{n\mu(n^2-1)\sin\theta_i\tan\theta_i}{\mu^2\sqrt{n^2\sin^2\theta_i - 1}}\frac{1}{1 - n^2/\mu^2}. \tag{2.5.47}$$

We are given the angle of incidence, and presumably the rhomb has $\mu \simeq 1$ (most materials do), so we can solve for n under this assumption and find

$$n = -\frac{\tan(5\pi/8)}{\sin(\theta_i)\sqrt{\tan^2\frac{5\pi}{8} - \tan^2\theta_i}}. \tag{2.5.48}$$

Substituting $\theta_i = 53.3°$ this becomes $n \approx 1.5$.

2.6 Radiation

Haters gonna hate. Accelerating charges gonna radiate.

That pretty much sums it up, but to be more precise, consider a point charge accelerating through space along a worldline $y^\mu(\tau)$. We're going to derive its electromagnetic field, called the Liénard-Wiechert field, and find the power it radiates at large distances. This is one of those times when shying away from relativity is a really bad idea, because relativity makes this problem much easier. The potential ϕ and vector potential $\boldsymbol{A}$ can be combined into a single four-vector $A^\mu = (\phi c, \boldsymbol{A})$. For a point charge at rest, this is given by

$$A^\mu = \frac{q}{r}(\partial_t)^\mu. \tag{2.6.1}$$

Now we generalize this to a particle with world-velocity λ^μ. Information about electromagnetic fields travels at the speed of light, so the field at x^μ should depend only on the data at the unique position y^μ on the worldline of the charge such that $(x-y)^\mu$ is a future-directed null vector. This is called the retarded time prescription.

Clearly we should replace $(\partial_t)^\mu$ by λ^μ evaluated at the retarded time. We would like to replace the distance r by the distance in a frame comoving with the charge, and since we are looking at the charge at a retarded time, we can equally well use the difference in time in a frame comoving with the charge, which is $-\lambda^\mu(x-y)_\mu$. So in total, we have

$$A^\mu = \frac{q}{r}\lambda^\mu, \tag{2.6.2}$$

where $r \equiv -\lambda^\nu(x-y)_\nu$ and everything is evaluated at the retarded time. This is the covariant expression for the Liénard-Wiechert field.

It is helpful to define a null vector field

$$k^\mu = \frac{(x-y)^\mu}{r}, \tag{2.6.3}$$

which points along light rays emitted on the worldline. It is normalized such that $k^\mu \partial_\mu k^\nu = 0$. With this and a few other identities, one can show that

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = \frac{q}{r}\left(k^\mu\left(\dot{\lambda}^\nu + \lambda^\nu\left(r^{-1} - k\cdot\dot{\lambda}\right)\right) - (\mu \leftrightarrow \nu)\right). \tag{2.6.4}$$

Now we can compute the power radiated by this field by integrating the Poynting vector. If we integrate over a sphere far from the particle, then only terms of order r^{-2} in the Poynting vector matter, which means that only terms of order r^{-1} in $F^{\mu\nu}$ matter. We may thus define

$$F^{\mu\nu}_{\text{rad}} = \frac{q}{r}\left(k^\mu\left(\dot{\lambda}^\nu - \lambda^\nu(k\cdot\dot{\lambda})\right) - (\mu \leftrightarrow \nu)\right). \tag{2.6.5}$$

This part of the field is proportional to $\dot{\lambda}$, as is to be expected. From a relativistic standpoint, the Poynting vector comes from the stress-energy tensor,

$$T^{\mu\nu} = \frac{1}{4\pi}\left(F^{\mu\alpha}F_\alpha{}^\nu - \frac{1}{4}\eta^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}\right). \tag{2.6.6}$$

The Poynting vector is the spatial part of $T^{\mu\nu}\lambda_\nu$. From (2.6.5), we have

$$T^{\mu\nu}_{\text{rad}}\lambda_\nu = \frac{q^2}{4\pi r^2}\left(\dot{\lambda}^\mu + (k^\mu - \lambda^\mu)(k\cdot\dot{\lambda}) + k^\mu\left((k\cdot\dot{\lambda})^2 - \dot{\lambda}\cdot\dot{\lambda}\right)\right). \tag{2.6.7}$$

At last we restrict our attention to a uniformly accelerated charge. In its instantaneous rest frame λ^μ is timelike and $k^\mu = (-1, -\hat{\boldsymbol{r}})$, so $k\cdot\dot{\lambda} = -\hat{\boldsymbol{r}}\cdot\boldsymbol{a}$. The Poynting vector is thus

$$\boldsymbol{S} = \frac{q^2}{4\pi r^2}\left(\boldsymbol{a} + \hat{\boldsymbol{r}}\left(-\hat{\boldsymbol{r}}\cdot\boldsymbol{a} + |\hat{\boldsymbol{r}}\times\boldsymbol{a}|^2\right)\right). \tag{2.6.8}$$

The total power radiated is

$$P = \frac{q^2}{2}\int_{-1}^{1} d(\cos\theta)\,\left(a^2\sin^2\theta\right). \tag{2.6.9}$$

The first term vanishes under integration, and we are left with

$$P = \frac{2q^2a^2}{3c^3}, \tag{2.6.10}$$

where we have restored factors of c by dimensional analysis. This the *Larmor formula* for the power radiated by an accelerating charge.

Problem 2.26 (M08E2)
A particle with charge e starting at rest is given uniform acceleration, a, for a time Δt to non-relativistic energies.

a) Compute the power radiated per unit solid angle by the electric charge as a function of the angle θ measured with respect to its direction of acceleration.

b) Assume the time Δt is infinitesimally short. Compute the total energy radiated per unit wavelength as a function of the final velocity v and wavelength λ of the radiated electromagnetic waves.

The answer to part (a) can be read off directly from the discussion above:

$$\frac{dP}{d\Omega} = \frac{e^2a^2}{4\pi c^3}\sin^2\theta. \tag{2.6.11}$$

The energy will be radiated by electromagnetic waves with some frequency spectrum, and we need to determine this frequency spectrum. The total energy is

$$E = \int dt\, d\Omega\, \frac{dP}{d\Omega} = \frac{2e^2}{3c^3}\int dt\, a(t)^2. \tag{2.6.12}$$

The Fourier transform of $a(t)$ is

$$f(\omega) = \int dt\, e^{i\omega t}a(t) = \frac{2a}{\omega}\sin\left(\frac{\omega\Delta t}{2}\right)e^{i\phi}, \tag{2.6.13}$$

where ϕ is some phase. Then substituting $a(t) = \frac{1}{2\pi}\int d\omega\, e^{-i\omega t}f(\omega)$ we have

$$E = \frac{4e^2a^2}{3\pi c^3}\int d\omega\, \frac{\sin^2\left(\frac{\omega\Delta t}{2}\right)}{\omega^2}, \tag{2.6.14}$$

and so

$$\frac{dE}{d\omega} = \frac{4e^2a^2}{3\pi c^3\omega^2}\sin^2\left(\frac{\omega\Delta t}{2}\right). \tag{2.6.15}$$

Since $\omega = \frac{2\pi c}{\lambda}$, this can be expressed as

$$\frac{dE}{d\lambda} = \frac{2e^2a^2}{3\pi^2c^4}\sin^2\left(\frac{\pi c\Delta t}{\lambda}\right). \tag{2.6.16}$$

Problem 2.27 (M00E2)
A particle of mass m and charge q is released from rest from a distance z_0 above an infinite grounded conducting plane. Neglect relativistic effects and gravity.

a) How long will it take for the particle to hit the plane? (Neglect radiation loss.) You may leave your answer in terms of a dimensionless integral.

b) What is the power radiated as a function of z?

Now consider the conducting plane to be replaced by a semi-infinite dielectric ϵ. (That is, for $z > 0$, there is a vacuum, and for $z < 0$, space is filled with the dielectric.)

c) Calculate the force on the charge q when it is a distance z_0 above the plane.

When the charge is at a height z, there will be an image charge at $-z$. Thus the total energy of the system is $-\frac{q^2}{4z_0^2}$, and so the velocity of the charge is

$$v = \left(\frac{q^2}{2m}\left(\frac{1}{z^2} - \frac{1}{z_0^2}\right)\right)^{1/2}. \tag{2.6.17}$$

The time to reach the plane is

$$T = \int_0^{z_0} \frac{dz}{v} = \frac{\sqrt{2m}z_0}{q}\int_0^{z_0}\left(\frac{z_0^2}{z^2} - 1\right)^{-1/2} dz. \tag{2.6.18}$$

Letting $u = (z/z_0)^2$, this is

$$T = \frac{\sqrt{2m}z_0^2}{2q}\int_0^1 (1-u)^{-1/2}\, du = \frac{\sqrt{2m}z_0^2}{q}. \tag{2.6.19}$$

The acceleration at z is $\frac{q^2}{4z^2m}$, so the power radiated is

$$P(z) = \frac{q^6}{24m^2z^4c^3}. \tag{2.6.20}$$

When $z < 0$ is filled with dielectric, there will be an image charge $\frac{1-\epsilon}{1+\epsilon}$ induced at $z = -z_0$. Thus, the force will be

$$\boldsymbol{F} = -\frac{1-\epsilon}{1+\epsilon}\frac{q^2}{4z_0^2}\hat{\boldsymbol{z}}. \tag{2.6.21}$$

Problem 2.28 (J06E3)
A classical particle of mass m and charge q moves in an isotropic three-dimensional harmonic potential with "spring constant" K such that its trajectory is nearly circular at all times.

a) What is the characteristic time (time constant) for the decay of the kinetic energy of this system due to electromagnetic radiation?

b) What condition(s) must be satisfied so that the fraction of the energy radiated per period of the motion is small (i.e. so that the the quality factor of this oscillator remains high), and hence the trajectory is indeed nearly circular?

c) Verify that this requirement implies that the radiation-reaction force is small compared to the spring force on the particle.

The acceleration of the particle is $\frac{KR}{m}$, where R is the radius of the circular orbit, and so it will radiate with power

$$P = \frac{2q^2}{3c^3}\frac{K^2R^2}{m^2}. \tag{2.6.22}$$

The kinetic energy is $\frac{1}{2}mv^2 = \frac{K}{2}R^2$, and so the time constant is

$$\tau = \frac{3m^2c^3}{4Kq^2}. \tag{2.6.23}$$

In order for the fraction of energy radiated per period to be small, the orbital period $2\pi\sqrt{\frac{m}{K}}$ should be much less than τ, or

$$\frac{3c^3}{8\pi q^2}\sqrt{\frac{m^3}{K}} \gg 1. \tag{2.6.24}$$

The radiation reaction force is $\frac{2q^2}{3c^3}\dot{a}$. Expressing a in terms of the energy as

$$a = \frac{KR}{m} = \sqrt{\frac{2EK}{m}}, \tag{2.6.25}$$

we have

$$\dot{a} = \sqrt{\frac{K}{2E}}P = \frac{2q^2}{3c^3}\frac{K^2R}{m^3}. \tag{2.6.26}$$

The radiation-reaction force is

$$F = \frac{2q^2}{3c^2}\dot{a} = \left(\frac{2q^2}{3c^3}\right)^2\frac{K^2R}{m^3}. \tag{2.6.27}$$

The force is KR, so for the radiation reaction force to be small compared to the spring force we must have

$$\left(\frac{3c^3}{2q^2}\right)^2\frac{m^3}{K} \gg 1, \tag{2.6.28}$$

which up to constants is the square of (2.6.24).

It's time to mention something that probably ought to have come up a bit earlier than this: the multipole expansion. First, as a warmup, let's look back at the electric potential of a static charge configuration,

$$\phi(\boldsymbol{r}) = \int \frac{\rho(\boldsymbol{r'})\,d\boldsymbol{r'}}{|\boldsymbol{r}-\boldsymbol{r'}|}. \tag{2.6.29}$$

Most of the time, when we have some charge, $\phi \sim \frac{1}{r}$. But when the total charge is zero, we've seen in several examples that $\phi \sim \frac{1}{r^2}$ instead; in this case, ϕ is proportional to the dipole moment. We can understand this in a very general way by expanding the denominator in the integral. We have

$$\phi(\boldsymbol{r}) = \int \frac{\rho(\boldsymbol{r'})\,d\boldsymbol{r'}}{\sqrt{\boldsymbol{r}^2+\boldsymbol{r'}^2-2\boldsymbol{r}\cdot\boldsymbol{r'}}} \tag{2.6.30}$$

$$= \frac{1}{r}\int \rho(\boldsymbol{r'})\left(1-2\frac{r'}{r}\cos\theta+\frac{(r')^2}{r^2}\right)^{-1/2} d\boldsymbol{r'} \tag{2.6.31}$$

$$= \frac{1}{r}\int \rho(\boldsymbol{r'})\left(1+\frac{r'}{r}\cos\theta+\frac{1}{2}\left(\frac{r'}{r}\right)^2\left(3\cos^2\theta-1\right)+\ldots\right) d\boldsymbol{r'}. \tag{2.6.32}$$

The first term gives $\frac{Q}{r}$, where Q is the total charge. This is the monopole field, and if it is nonzero it dominates at large r. The next-to-leading field is

$$\frac{\int \rho(\boldsymbol{r'}) r' \cos\theta \, d\boldsymbol{r'}}{r^2} = \frac{\boldsymbol{p}\cdot\hat{\boldsymbol{r}}}{r^2}, \tag{2.6.33}$$

where $\boldsymbol{p} = \int \rho(\boldsymbol{r'})\boldsymbol{r'}\, d\boldsymbol{r'}$ is the dipole moment. For the quadrupole field, we define a tensor

$$Q_{ij} = \frac{1}{2}\int \rho(\boldsymbol{r'}) \left(3r'_i r'_j - \delta_{ij}\right)\, d\boldsymbol{r'}, \tag{2.6.34}$$

so that the field becomes $\frac{Q_{ij}\hat{r}_i\hat{r}_j}{r^3}$. In general, the expansion of the denominator gives Legendre polynomials in $\cos\theta$, and integrating order by order gives the multipole expansion of the electric potential.

So much for that. It turns out that radiation works in a similar but somewhat more complicated way. We start with something that, again, probably should have been mentioned before (but also it's sort of obvious when you think about it): the electric and magnetic potentials in the general case are the integrals of the sources at retarded time. That is,

$$\phi(\boldsymbol{r},t) = \int \frac{\rho\left(\boldsymbol{r'}, t - \frac{|\boldsymbol{r}-\boldsymbol{r'}|}{c}\right)}{|\boldsymbol{r}-\boldsymbol{r'}|}\, d\boldsymbol{r'}, \qquad \boldsymbol{A}(\boldsymbol{r},t) = -\frac{1}{c}\int \frac{\boldsymbol{j}\left(\boldsymbol{r'}, t - \frac{|\boldsymbol{r}-\boldsymbol{r'}|}{c}\right)}{|\boldsymbol{r}-\boldsymbol{r'}|}\, d\boldsymbol{r'}. \tag{2.6.35}$$

Note that we're making a gauge choice (Coulomb gauge, $\nabla\cdot\boldsymbol{A} = 0$) in order to write the fields this way. If we have general, wibbly wobbly sources, it will be hard to integrate out the fields (not to mention the sadistic complications which come from including back-reactions). So, let's make a simplifying assumption $\rho(\boldsymbol{r},t) = \rho(\boldsymbol{r})e^{i\omega t}$, and similarly for $\boldsymbol{j}$. Then the retarded time prescription turns into

$$\phi(\boldsymbol{r},t) = e^{i\omega t}\int \frac{\rho\left(\boldsymbol{r'}\right) e^{-ik|\boldsymbol{r}-\boldsymbol{r'}|}}{|\boldsymbol{r}-\boldsymbol{r'}|}\, d\boldsymbol{r'}, \qquad \boldsymbol{A}(\boldsymbol{r},t) = -\frac{e^{i\omega t}}{c}\int \frac{\boldsymbol{j}\left(\boldsymbol{r'}\right) e^{-ik|\boldsymbol{r}-\boldsymbol{r'}|}}{|\boldsymbol{r}-\boldsymbol{r'}|}\, d\boldsymbol{r'}. \tag{2.6.36}$$

It's tempting to expand the denominator like before, but this would be a mistake. This is radiation, so we're free to take $r \gg r'$ and focus on the far field. Then the leading variations will come from the exponential, the parameter of which varies on the scale of r'/λ, where $k = \omega/c = 2\pi/\lambda$. In summary, we assume $r' \ll \lambda \ll r$ and pull $|\boldsymbol{r}-\boldsymbol{r'}| \sim r^{-1}$ out of the integral. To deal with the exponential, we approximate $|\boldsymbol{r}-\boldsymbol{r'}| \approx r - \frac{\boldsymbol{r'}\cdot\boldsymbol{r}}{r}$, so we have

$$\phi(\boldsymbol{r},t) = \frac{e^{i(\omega t - kr)}}{r}\int \rho\left(\boldsymbol{r'}\right) e^{ik\hat{\boldsymbol{r}}\cdot\boldsymbol{r'}}\, d\boldsymbol{r'} \qquad \boldsymbol{A}(\boldsymbol{r},t) = -\frac{e^{i(\omega t - kr)}}{rc}\int \boldsymbol{j}\left(\boldsymbol{r'}\right) e^{ik\hat{\boldsymbol{r}}\cdot\boldsymbol{r'}}\, d\boldsymbol{r'}. \tag{2.6.37}$$

Now we can expand the exponentials and look at the potentials term-by-term to find the multipole contributions to the radiation field. Expanding two different integrals sounds annoying, so we start by noting that far from the sources, $\frac{\partial \boldsymbol{E}}{\partial t} = c\nabla\times\boldsymbol{B}$. Since both fields have $e^{i\omega t}$ time dependence, this gives

$$\boldsymbol{E} = \frac{c}{i\omega}\nabla\times\boldsymbol{B} = \frac{1}{ik}\nabla\times(\nabla\times\boldsymbol{A}). \tag{2.6.38}$$

This means we can ignore ϕ altogether and derive everything from the expansion of $\boldsymbol{A}$. Let's start with the first term,

$$\boldsymbol{A}_1 = -\frac{e^{i(\omega t - kr)}}{rc}\int \boldsymbol{j}\left(\boldsymbol{r'}\right)\, d\boldsymbol{r'}. \tag{2.6.39}$$

Integrating by parts, this becomes

$$\boldsymbol{A}_1 = \frac{e^{i(\omega t - kr)}}{rc} \int \boldsymbol{r'} \left(\nabla \cdot \boldsymbol{j}\left(\boldsymbol{r'}\right)\right) \, d\boldsymbol{r'}. \tag{2.6.40}$$

Using the conservation of charge $\frac{\partial \rho}{\partial t} + \nabla \cdot \boldsymbol{j} = 0$, and recalling $\rho(\boldsymbol{r}, t) = e^{i\omega t}\rho(\boldsymbol{r})$, this becomes

$$\boldsymbol{A}_1 = -\frac{ike^{i(\omega t - kr)}}{r} \int \rho\left(\boldsymbol{r'}\right) \boldsymbol{r'} \, d\boldsymbol{r'} = -\frac{ik}{r}\boldsymbol{p}\left(t - \frac{r}{c}\right). \tag{2.6.41}$$

This term is thus called the dipole radiation term. As a quick check, we compute the fields

$$\boldsymbol{B}_1 = \nabla \times \boldsymbol{A} = -\frac{k^2}{r}\left(\hat{\boldsymbol{r}} \times \boldsymbol{p}\left(t - \frac{r}{c}\right)\right) + \mathcal{O}\left(r^{-2}\right), \qquad \boldsymbol{E}_1 = \frac{1}{ik}\nabla \times \boldsymbol{B}_1 = \frac{k^2}{r}\left(\hat{\boldsymbol{r}} \times \left(\hat{\boldsymbol{r}} \times \boldsymbol{p}\left(t - \frac{r}{c}\right)\right)\right). \tag{2.6.42}$$

The Poynting vector is

$$\boldsymbol{S}_1 = \frac{c}{4\pi}\boldsymbol{E}_1 \times \boldsymbol{B}_1 = \frac{\omega^4 |\boldsymbol{p}|^2}{4\pi r^2 c^3} \sin^2\theta \hat{\boldsymbol{r}}, \tag{2.6.43}$$

so the total power radiated is $P_1 = \frac{\omega^4}{3c^3}|\boldsymbol{p}|^2$. Indeed, if we model the dipole as a charge q oscillating between $z = \pm\ell$ with frequency ω, the Larmor formula gives

$$P = \frac{2q^2}{3c^3}\langle a^2 \rangle = \frac{q^2\ell^2\omega^4}{3c^3}, \tag{2.6.44}$$

and since $p = q\ell$ the two results agree.

Cool. So, next term. It's harder, but that's life. We have

$$\boldsymbol{A}_2 = -\frac{ike^{i(\omega t - kr)}}{rc} \int \boldsymbol{j}\left(\boldsymbol{r'}\right)\left(\hat{\boldsymbol{r}} \cdot \boldsymbol{r'}\right) \, d\boldsymbol{r'}. \tag{2.6.45}$$

Let's meditate on the integrand a bit. It looks like $\boldsymbol{j}$ weighted by $\cos\theta$, where θ has the obvious meaning. This is sort of like the electric dipole moment, where ρ is weighted by $\cos\theta$, except it's $\boldsymbol{j}$, so maybe it's a magnetic dipole moment. Indeed, the magnetic dipole moment is supposed to be IA for a current loop enclosing area A, and we recover that via Stokes' theorem if we define

$$\boldsymbol{\mu} = \frac{1}{2}\int \boldsymbol{r'} \times \boldsymbol{j}\left(\boldsymbol{r'}\right) \, d\boldsymbol{r'}. \tag{2.6.46}$$

This won't quite do, since our integrand is proportional to $\boldsymbol{j}$. But we have

$$\hat{\boldsymbol{r}} \times \left(\boldsymbol{r'} \times \boldsymbol{j}\right) = \left(\boldsymbol{j} \cdot \hat{\boldsymbol{r}}\right)\boldsymbol{r'} - \boldsymbol{j}\left(\hat{\boldsymbol{r}} \cdot \boldsymbol{r'}\right), \tag{2.6.47}$$

and voilá, the second term on the right is the integrand (with a sign). It follows that we can express the integrand as

$$\boldsymbol{j}\left(\boldsymbol{r'}\right)\left(\hat{\boldsymbol{r}} \cdot \boldsymbol{r'}\right) = -\frac{1}{2}\hat{\boldsymbol{r}} \times \left(\boldsymbol{r'} \times \boldsymbol{j}\right) + \frac{1}{2}\left(\left(\boldsymbol{j} \cdot \hat{\boldsymbol{r}}\right)\boldsymbol{r'} + \boldsymbol{j}\left(\hat{\boldsymbol{r}} \cdot \boldsymbol{r'}\right)\right). \tag{2.6.48}$$

The first term is related to the magnetic dipole, and (spoiler) the second term is related to the electric quadrupole.

Indeed, the integral coming from the first term is

$$\boldsymbol{A}_{2m} = \frac{ike^{i(\omega t - kr)}}{rc}\hat{\boldsymbol{n}} \times \frac{1}{2}\int \boldsymbol{r'} \times \boldsymbol{j}\left(\boldsymbol{r'}\right) \, d\boldsymbol{r'} = \frac{ik}{rc}\hat{\boldsymbol{r}} \times \boldsymbol{\mu}\left(t - \frac{r}{c}\right). \tag{2.6.49}$$

This looks a lot like the electric dipole field, which is nice. The fields are

$$\boldsymbol{B}_{2m} = -\frac{k^2}{rc}\hat{\boldsymbol{r}} \times \left(\hat{\boldsymbol{r}} \times \boldsymbol{\mu}\left(t - \frac{r}{c}\right)\right), \qquad \boldsymbol{E}_{2m} = \hat{\boldsymbol{r}} \times \boldsymbol{B}_{2m}. \tag{2.6.50}$$

This gives a Poynting vector of

$$\boldsymbol{S}_{2m} = \frac{c}{4\pi}\boldsymbol{E}_{2m} \times \boldsymbol{B}_{2m} = \frac{k^4|\boldsymbol{\mu}|^2}{4\pi r^2 c}\sin^2\theta\hat{\boldsymbol{r}}, \tag{2.6.51}$$

and so the total power is $P_{2m} = \frac{\omega^4\mu^2}{3c^5}$.

The multipole expansion is only useful if it gives terms which decrease in intensity, so we should compare P_{2m} to P_1. To make a good comparison, we fix the current $j \sim \omega p \sim \frac{\mu}{a}$. Then

$$\frac{P_{2m}}{P_1} = \frac{\mu^2}{c^2p^2} \sim \left(\frac{a}{\lambda}\right)^2. \tag{2.6.52}$$

We have already assumed $a \ll \lambda$, so the expansion is indeed a good one.

To finish off, we get the electric quadrupole field from the other term in our integrand for $\boldsymbol{A}_2$. We have

$$\boldsymbol{A}_{2q} = -\frac{ike^{i(\omega t - kr)}}{2rc}\int \left(\boldsymbol{j}\left(\hat{\boldsymbol{r}} \cdot \boldsymbol{r'}\right) + \boldsymbol{r'}\left(\boldsymbol{j} \cdot \hat{\boldsymbol{r}}\right)\right) d\boldsymbol{r'}. \tag{2.6.53}$$

By writing

$$j_k\hat{r}_j r'_j = \nabla'_i\left(j_i\hat{r}_j r'_j r'_k\right) - r'_k\hat{r}_j r'_j\nabla'_i j_i - r'_k j_i\nabla'_i\left(\hat{r}_j r'_j\right), \tag{2.6.54}$$

we find

$$\int \boldsymbol{j}\left(\hat{\boldsymbol{r}} \cdot \boldsymbol{r'}\right) d\boldsymbol{r'} = -\int \boldsymbol{r'}\left(\hat{\boldsymbol{r}} \cdot \boldsymbol{r'}\right)\left(\nabla' \cdot \boldsymbol{j}\right) d\boldsymbol{r'} - \int \boldsymbol{r'}\left(\boldsymbol{j} \cdot \hat{\boldsymbol{r}}\right) d\boldsymbol{r'}. \tag{2.6.55}$$

Substituting this into $\boldsymbol{A}_{2q}$ and using $\nabla \cdot \boldsymbol{j} = -\frac{\partial\rho}{\partial t}$, there is a convenient cancellation and we find

$$\boldsymbol{A}_{2q} = \frac{k^2e^{i(\omega t - kr)}}{2r}\int \boldsymbol{r'}\left(\hat{\boldsymbol{r}} \cdot \boldsymbol{r'}\right)\rho\left(\boldsymbol{r'}\right) d\boldsymbol{r'}. \tag{2.6.56}$$

This shows that we are dealing with something "electric," but what about "quadrupole"? Well, remember that $\boldsymbol{A}_{2q}$ is only defined up to a gauge, and also, we only care about fields at order $\frac{1}{r}$. If we add a vector proportional to $\hat{\boldsymbol{r}}$ to $\boldsymbol{A}_{2q}$, its curl will be unchanged at order r^{-1}. So, we can equally well write

$$\boldsymbol{A}_{2q} = \frac{k^2e^{i(\omega t - kr)}}{3r}\hat{r}^i\int \frac{3r'_i r'_j - \delta_{ij}}{2}\rho\left(\boldsymbol{r'}\right) d\boldsymbol{r'} = \frac{k^2e^{i(\omega t - kr)}}{3r}\hat{r}^i Q_{ij}, \tag{2.6.57}$$

where Q_{ij} is the electric quadrupole tensor.

All these fortuitous cancellations, gauge transformations, etc., are the result of a much deeper mathematical fact. Even though we did a very different sort of expansion in the radiation case, the fields are essentially forced to organize themselves into representations of the rotation group $SO(3)$, and this is the heart and soul of the multipole expansion. But enough of this for now.

Problem 2.29 (J08E1)
An antenna consists of a circular wire loop of radius R, centered in the xy plane of a Cartesian coordinate system. The current has the same amplitude, $I = I(t)$, at all locations in the wire at a given time t. There is no net electrical charge on the wire. Assuming that $\dot{I}$, the rate of change of the current, is slow enough that magnetic dipole radiation dominates any higher multipoles, calculate:

a) the vector potential $\boldsymbol{A} = \boldsymbol{A}(\boldsymbol{r}, t)$ and scalar potential ϕ at the location $\boldsymbol{r}$ and time t when $r \gg cI/\dot{I}$ (specify your choice of gauge);

b) the magnetic and electric fields, $\boldsymbol{B}$ and $\boldsymbol{E}$, at $\boldsymbol{r}$ and t;

c) the energy flux, $S = S(\theta, \phi)$, as a function of the polar angles θ and ϕ;

d) the total radiated power $\mathrm{P} = \int S \sin\theta \, d\theta \, d\phi$.

Retain enough terms of any expansion in powers of $1/r$ to account for radiation. Insofar as possible, express your answers in terms of the magnetic dipole moment, $m = \pi R^2 I/c$, and its time derivatives.

There is no net charge density on the wire, so ϕ vanishes. The vector potential for a magnetic dipole is

$$\boldsymbol{A}_{2m} = \frac{ik}{rc}\hat{\boldsymbol{r}} \times \boldsymbol{u}\left(t - \frac{r}{c}\right) = \frac{1}{rc^2}\hat{\boldsymbol{r}} \times \dot{\boldsymbol{\mu}}\left(t - \frac{r}{c}\right), \tag{2.6.58}$$

where we have used $i\omega = \frac{\dot{\mu}}{\mu}$. The fields up to order r^{-1} are then

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}_{2m} = -\frac{ik}{rc^2}\hat{\boldsymbol{r}} \times (\hat{\boldsymbol{r}} \times \dot{\boldsymbol{\mu}}) = \frac{1}{rc^3}\left(\ddot{\boldsymbol{\mu}} - \hat{\boldsymbol{r}}(\ddot{\boldsymbol{\mu}} \cdot \hat{\boldsymbol{r}})\right), \tag{2.6.59}$$

$$\boldsymbol{E} = \frac{1}{ik}\nabla \times \boldsymbol{B} = -\frac{1}{rc^3}\hat{\boldsymbol{r}} \times \left(\ddot{\boldsymbol{\mu}} - \hat{\boldsymbol{r}}(\ddot{\boldsymbol{\mu}} \cdot \hat{\boldsymbol{r}})\right), \tag{2.6.60}$$

where $\boldsymbol{\mu}$ is always evaluated at retarded time.

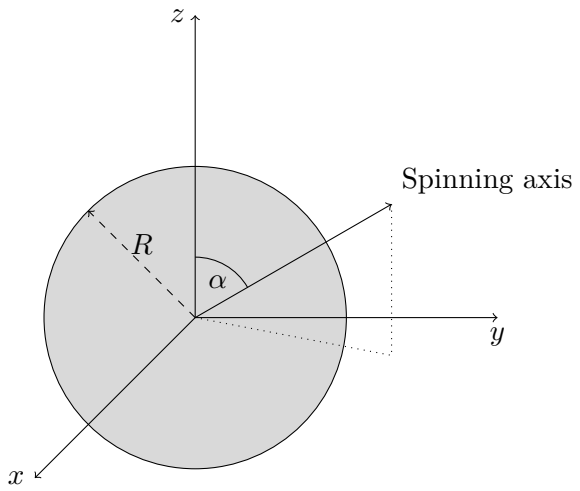
The energy flux is

$$S(\theta, \phi) = \boldsymbol{S} \cdot \hat{\boldsymbol{r}} = \frac{1}{4\pi r^2 c^6}\ddot{\mu}^2 \sin^2\theta. \tag{2.6.61}$$

This gives a total radiated power

$$P = \frac{2\ddot{\mu}^2}{3c^6}. \tag{2.6.62}$$

Problem 2.30 (M99E1)
Electromagnetic radiation of wavelength λ is observed to originate from a system consisting of an electrically charged sphere of radius R placed in a uniform magnetic field B and spinning about its axis with a very large angular velocity ω. The spin axis of the sphere, which is free to move, makes an angle α with the field direction. Assume $R \ll \lambda$.



a) Explain briefly why the system radiates electromagnetic energy.

b) Find in terms of the given quantities, not all of which may be necessary, the ratio Q/M of the total charge Q to the mass M of the sphere assuming that both charge and mass are uniformly distributed over its volume.

c) What is the polarization of the radiation field?

The system radiates because it precesses. When the spin axis is displaced at angle α from the magnetic field, there is a resulting torque $\boldsymbol{m} \times \boldsymbol{B}$ on the magnetic moment $\boldsymbol{m}$ formed due to the spin of the sphere. Since $\boldsymbol{m} \propto \boldsymbol{L}_s$ (the spin angular momentum), this torque keeps $\boldsymbol{m}$ spinning around $\boldsymbol{B}$. Since the magnetic moment is changing, we get magnetic dipole radiation.

In more detail, let $\boldsymbol{m} = g\boldsymbol{L}_s$ (we'll compute g later). Then

$$\frac{d\boldsymbol{m}}{dt} = g\frac{d\boldsymbol{L}_s}{dt} = g\boldsymbol{m} \times \boldsymbol{B}. \tag{2.6.63}$$

This implies that gB is the precession frequency of the sphere. The wavelength of radiation should be $\lambda = 2\pi/k = 2\pi c/\omega$, and so we solve and find

$$g = \frac{2\pi c}{\lambda B}. \tag{2.6.64}$$

It remains to determine g in terms of Q and M. The angular momentum is

$$L_s = I\omega = \frac{2}{5}MR^2\omega. \tag{2.6.65}$$

The magnetic moment is a bit trickier. The charge density is $\rho = \frac{3Q}{4\pi R^3}$. At radius r and angle θ, we have a current loop with $I = (r\rho\, dr\, d\theta)(\omega r \sin\theta)$ and area $\pi r^2 \sin^2\theta$, so the total magnetic moment magnitude is

$$m = \pi\rho\omega \int_0^R \int_0^\theta r^4 \sin^3\theta\, d\theta\, dr = \frac{2\pi\rho\omega R^5}{15}. \tag{2.6.66}$$

Thus, $g = \frac{\pi\rho R^3}{3M} = \frac{Q}{4M}$, and hence

$$\frac{Q}{M} = \frac{8\pi c}{\lambda B}. \tag{2.6.67}$$

Since the radiation is due to the changing magnetic dipole, the fields look just as they did in the previous problem, where now $\ddot{\boldsymbol{\mu}} \propto -(\cos\psi\hat{\boldsymbol{x}} - \sin\psi\hat{\boldsymbol{y}})$. An observer at a point on one of the poles will see circularly polarized light. Anywhere else, the light will be elliptically polarized in some more complicated pattern.

2.7 Additional Problems

Problem 2.31 (J01E2)
A betatron is a device in which ultrarelativistic electrons are held in a circle of fixed radius R (taken to be centered on the origin in the xy plane) by a magnetic field $B_z(r,t)$ while their energy is increased via a changing magnetic flux $d\Phi/dt = \pi R^2 dB_{\text{avg},z}/dt$ through the circle. Motion of the electrons perpendicular to the circle is prevented by means that need not be considered here.

Deduce the relation between the magnetic field B_z at radius R and the magnetic field $B_{\text{avg},z}$ averaged over the area of the circle. Also deduce the maximum energy E to which an electron could be accelerated by a betatron in terms of B_z, $dB_{\text{avg},z}/dt$ and R.

We have

$$\gamma\frac{mv^2}{R} = e\frac{v}{c}B, \tag{2.7.1}$$

so

$$v = \frac{eBR}{\gamma mc}. \tag{2.7.2}$$

The electrons are accelerated by the induced electric force, so

$$\gamma m\frac{dv}{dt} = \frac{e\pi R^2}{2\pi R}\frac{1}{c}\frac{dB_{\text{avg},z}}{dt}. \tag{2.7.3}$$

Comparing these two equations, we have

$$\frac{dB}{dt} = \frac{\gamma mc}{eR}\frac{dv}{dt} = \frac{1}{2}\frac{dB_{\text{avg},z}}{dt}. \tag{2.7.4}$$

If we assume the system starts at $B = B_{\text{avg},z} = 0$, then $B = \frac{1}{2}B_{\text{avg},z}$ for all times in order to keep the electrons in a circle.

It would seem as though we can deposit an arbitrary amount of energy into the electrons, except that they will also radiate with

$$P = \frac{2e^2}{3c^2}\left(\gamma^2\frac{v^2}{R}\right)^2 = \frac{2e^6B^4R^2}{3m^4c^7}. \tag{2.7.5}$$

Thus, the electron cannot increase in energy once the energy input from the induced electric field reaches this value. The energy input is

$$\frac{dE}{dt} = v\frac{e\pi R^2}{2\pi R}\frac{1}{c}\frac{dB_{\text{avg},z}}{dt} = \frac{e^2BR^2}{2\gamma mc^2}\frac{dB_{\text{avg},z}}{dt}. \tag{2.7.6}$$

Setting these equal we find

$$E = \gamma mc^2 = \frac{3m^4c^7}{4e^4B^3}\frac{dB_{\text{avg},z}}{dt}. \tag{2.7.7}$$

Problem 2.32 (M12E2)
Thomson scattering is the scattering of light from a free electron (no binding, no damping). Derive the total cross section σ for Thomson scattering by treating the scattered radiation generated by an electron in an electromagnetic plane wave. Assume that the speed of the electron remains small: $v \ll c$.

First we need to determine how the electron moves in the plane wave. Take the fields to be

$$\boldsymbol{E} = \boldsymbol{E}_0 e^{i(kz-\omega t)}, \qquad \boldsymbol{B} = \hat{\boldsymbol{z}} \times \boldsymbol{E}. \tag{2.7.8}$$

Assume the electron starts at rest at the origin at $t = 0$. The Lorentz force is

$$m\frac{d\boldsymbol{v}}{dt} = e\exp\left(i(kz-\omega t)\right)\left(\boldsymbol{E}_0 + \frac{\boldsymbol{v}}{c} \times (\hat{\boldsymbol{z}} \times \boldsymbol{E}_0)\right). \tag{2.7.9}$$

Since $v \ll c$, we can ignore the magnetic force. The average power radiated is thus

$$\langle P \rangle = \frac{2e^2}{3c^3}\left\langle \left(\text{Re}\,\frac{d\boldsymbol{v}}{dt}\right)^2 \right\rangle = \frac{e^4}{3m^2c^3}E_0^2. \tag{2.7.10}$$

The Poynting vector of the plane wave is $\frac{c}{4\pi}E_0^2\hat{\boldsymbol{z}}$, so this much power comes from an area

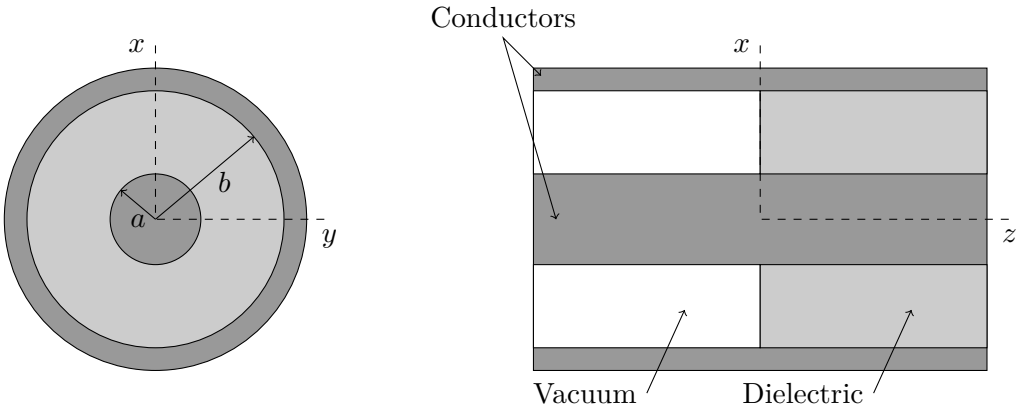
$$\sigma = \frac{4\pi e^4}{m^2c^4}. \tag{2.7.11}$$

Problem 2.33 (J11E1)
Electromagnetic radiation in TEM mode propagates along a coaxial waveguide consisting of two concentric, right circular cylinders of infinite conductivity. The radius of the inner conductor is a, and of the inner surface of the outer conductor is b. The region of the conductors corresponding to negative values of z (the symmetry axis of the cylinders) is vacuum. The region at positive z is filled with a uniform lossless dielectric, (dielectric constant ϵ). The wave is propagating only in the positive direction in the dielectric, while there are incident and reflected waves in the vacuum region. Assume the incident wave has peak electric field E_0 at the surface of the inner conductor, and oscillates with frequency ω.

a) In terms of E_0, a, b and any necessary constants for your system of units, find the time-averaged power of the incident wave propagating in the vacuum side of the cable.

b) Calculate the electric and magnetic fields for the reflected, and transmitted waves. Specify the amplitudes, and space and time dependence.

c) Find the average force per unit area on the dielectric interface at $z = 0$.



We are given that the electric and magnetic fields are transverse to the direction of propagation, and we know that electric fields are normal to the surfaces of conductors, so $\boldsymbol{E}$ must point in the radial direction. In order to be divergence free, it must fall off as $\frac{1}{r}$. This fixes

$$\boldsymbol{E} = \frac{E_0 a}{r} e^{i\omega(z/c-t)} \hat{\boldsymbol{r}}, \qquad \boldsymbol{B} = \frac{E_0 a}{r} e^{i\omega(z/c-t)} \hat{\boldsymbol{\theta}}, \tag{2.7.12}$$

and so the Poynting vector is

$$\langle \boldsymbol{S} \rangle = \frac{c}{4\pi} \langle \boldsymbol{E} \times \boldsymbol{B} \rangle = \frac{E_0^2 a^2 c}{8\pi r^2} \hat{\boldsymbol{z}}. \tag{2.7.13}$$

The total power is then

$$P = \frac{E_0^2 a^2 c}{4} \log \frac{b}{a}. \tag{2.7.14}$$

To find the reflected and transmitted waves, we can think about some limiting cases. If $\epsilon = 1$, everything is transmitted. If $\epsilon = \infty$, i.e. the dielectric is actually a conductor, then nothing is transmitted and the reflected wave electric field is the negative of the incoming wave electric field in order to satisfy the boundary condition. Furthermore, the reflection and transmission coefficients will depend on the index of refraction, $n = \sqrt{\epsilon}$. From these considerations, it sounds like the reflected wave should be $\frac{1-\sqrt{\epsilon}}{1+\sqrt{\epsilon}}$ times the incoming wave, and then conservation of energy requires that the transmitted wave is $\frac{2}{1+\sqrt{\epsilon}}$ times the incoming wave. Explicitly, we have

$$\boldsymbol{E}_r = \frac{1-\sqrt{\epsilon}}{1+\sqrt{\epsilon}} \frac{E_0 a}{r} e^{i\omega(-z/c-t)} \hat{\boldsymbol{r}}, \qquad \boldsymbol{B}_r = -\frac{1-\sqrt{\epsilon}}{1+\sqrt{\epsilon}} \frac{E_0 a}{r} e^{i\omega(-z/c-t)} \hat{\boldsymbol{\theta}}, \tag{2.7.15}$$

$$\boldsymbol{E}_t = \frac{2}{1+\sqrt{\epsilon}} \frac{E_0 a}{r} e^{i\omega(z/c-t)} \hat{\boldsymbol{r}}, \qquad \boldsymbol{B}_t = \frac{2}{1+\sqrt{\epsilon}} \frac{E_0 a}{r} e^{i\omega(z/c-t)} \hat{\boldsymbol{\theta}}. \tag{2.7.16}$$

The force on the interface comes from the change in the momentum of the reflected light, so we have

$$2\frac{|\boldsymbol{S}_r \cdot \hat{\boldsymbol{n}}|}{c} = \frac{E_0^2 a^2}{4\pi r^2}\frac{(1-\sqrt{\epsilon})^2}{(1+\sqrt{\epsilon})^2}. \tag{2.7.17}$$

The average pressure is then

$$P_{\text{avg}} = \frac{E_0^2 a^2}{2\pi(b^2-a^2)}\frac{(1-\sqrt{\epsilon})^2}{(1+\sqrt{\epsilon})^2}\log\frac{b}{a}. \tag{2.7.18}$$

Chapter 3

Quantum Mechanics

Little known historical fact: Planck and de Broglie and Schrödinger and Heisenberg and friends were all big Dr. Seuss fans. One of their favorites was *Wacky Wednesday*. One day, they all decided to write some *Wacky Wednesday* fan fiction together. Planck started it off: what if light was organized into quanta of energy? This would be wacky and also solve the ultraviolet catastrophe. de Broglie got a bit wackier by positing a wave-particle duality and noting that this would naturally quantize electron orbits around a nucleus. Schrödinger and Heisenberg really got things going when they wrote down their wave and matrix mechanics, respectively. Now that is some wacky stuff.



Figure 3.1: (Left) Max Planck, who suggested in 1900 that discretizing the energy of light into packets called *quanta* could solve the ultraviolet catastrophe in the Rayleigh-Jeans law, and who also wants you to get off his damn lawn. (Center) Erwin Schrödinger, who still holds the Guinness World Record for Most Circular Glasses, and who expressed quantum mechanics in terms of wavefunctions. (Right) Werner Heisenberg, who's too cool for school, and who wrote down quantum mechanics in terms of matrices, a formulation which turned out to be equivalent to Schrödinger's.

In 3.1, we'll review operator formalism, the Schrödinger equation, all that good stuff. In 3.2 we'll look at the quantum harmonic oscillator all its glory. In 3.3 we'll reapply some of the same methods to look at spin and angular momentum. In 3.4 we'll give up on getting things exactly right and do perturbation theory. In 3.5 we'll perturb things in a time-dependent way, and in 3.6 we'll bounce things off each other.

3.1 Review of Basics

We start from the Hamiltonian formulation of mechanics, in which the equations of motion are

$$\frac{dx^i}{dt} = \frac{\partial H}{\partial p_i}, \qquad \frac{dp^i}{dt} = -\frac{\partial H}{\partial x_i}. \tag{3.1.1}$$

This leads to a nice way to write the time derivative of any quantity:

$$\frac{dA}{dt} = \frac{\partial A}{\partial x^i}\frac{\partial H}{\partial p_i} - \frac{\partial A}{\partial p^i}\frac{\partial H}{\partial x_i} + \frac{\partial A}{\partial t} = \{A, H\} + \frac{\partial A}{\partial t}, \tag{3.1.2}$$

where we have defined the Poisson bracket

$$\{A, B\} = \frac{\partial A}{\partial x^i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p^i}\frac{\partial B}{\partial x_i}. \tag{3.1.3}$$

Note that $\{x^i, p_j\} = \delta^i_j$.

The Poisson bracket gives a Lie algebra structure to classical quantities. We quantize a classical theory by attempting to form a representation of this algebra $A \mapsto \hat{A}$ such that

$$[\hat{A}, \hat{B}] = i\hbar\{A, B\}. \tag{3.1.4}$$

Note in particular the canonical commutation relations,

$$[\hat{x}^i, \hat{p}_j] = i\hbar\delta^i_j. \tag{3.1.5}$$

Equations of motion for these operators in the Heisenberg representation are given by lifting the equation above to read

$$i\hbar\frac{d\hat{A}}{dt} = [\hat{A}, \hat{H}] + i\hbar\frac{\partial \hat{A}}{\partial t}. \tag{3.1.6}$$

We see that the operator $\hat{H}$ determines the time dependence of all other operators which are not explicitly time dependent.

Let's assume $\hat{A}$ has no explicit time dependence. Then the matrix elements of this operator change according to

$$i\hbar\frac{d}{dt}\left(\langle\psi|\hat{A}|\phi\rangle\right) = \langle\psi|[\hat{A}, \hat{H}]|\phi\rangle = \langle\psi|\hat{A}\hat{H}|\phi\rangle - \langle\psi|\hat{H}\hat{A}|\phi\rangle\,. \tag{3.1.7}$$

This would also be satisfied if we took $\hat{A}$ to be constant and imposed a time dependence on the states by

$$i\hbar\frac{d}{dt}\,|\psi\rangle = \hat{H}\,|\psi\rangle\,. \tag{3.1.8}$$

This is Schrödinger's equation. We can solve it by decomposing $|\psi\rangle$ into eigenstates of $\hat{H}$, for which the equation is trivial to solve. Thus, the real meat of solving the equation comes in solving

$$\hat{H}\,|\psi\rangle = E\,|\psi\rangle\,, \tag{3.1.9}$$

sometimes called the time-independent Schrödinger equation.

Problem 3.1 (M07Q1)
Let $F(\boldsymbol{r}, \boldsymbol{p})$ be some function of position and momentum without explicit time dependence, $\frac{\partial F}{\partial t} = 0$.

a) If $|\psi_n\rangle$ is an eigenstate of a Hamiltonian $\hat{H}$ in the Schrödinger representation, show that

$$\frac{d}{dt}\left(\langle\psi_n \,|\, F \,|\, \psi_n\rangle\right) = 0. \tag{3.1.10}$$

b) Suppose

$$H = \frac{\boldsymbol{p}^2}{2m} + V(\boldsymbol{r}). \tag{3.1.11}$$

Show that

$$\left\langle \psi_n \left| \frac{\boldsymbol{p}^2}{2m} \right| \psi_n \right\rangle = \frac{1}{2}\left\langle \psi_n \,|\, \boldsymbol{r}\cdot\nabla V(\boldsymbol{r}) \,|\, \psi_n\right\rangle. \tag{3.1.12}$$

c) Use this quantum-mechanical version of the virial theorem to estimate the fraction of the proton rest mass that is in the form of potential energy. The gluon-mediated force between two quarks is nearly independent of the distance between them. The rest mass of the quarks is much smaller than the mass of the proton, so strictly speaking one should use a relativistic version of the virial theorem. However, the non-relativistic version still gives approximately correct results, see "Relativistic virial theorem", Phys. Rev. Lett. 64, 2733-2735 (1990).

We have

$$\frac{d}{dt}\left(\langle\psi_n \,|\, F \,|\, \psi_n\rangle\right) = -\frac{i}{\hbar}\left\langle \psi_n \,|\, [F, H] \,|\, \psi_n\right\rangle, \tag{3.1.13}$$

and since $|\psi_n\rangle$ is an eigenstate of H, the expectation of this commutator vanishes.

Let

$$F(\boldsymbol{r}, \boldsymbol{p}) = \boldsymbol{r}\cdot\boldsymbol{p}. \tag{3.1.14}$$

Then

$$0 = \frac{d}{dt}\left(\langle\psi_n \,|\, F \,|\, \psi_n\rangle\right) = \left\langle \psi_n \,|\, (\boldsymbol{v}\cdot\boldsymbol{p} - \boldsymbol{r}\cdot\nabla\boldsymbol{V}(\boldsymbol{r})) \,|\, \psi_n\right\rangle, \tag{3.1.15}$$

from which the result follows.

If we have a force independent of distance, then $V(\boldsymbol{r}) \sim |\boldsymbol{r}|$, and so the virial theorem in this form gives

$$\left\langle \psi_n \left| \frac{\boldsymbol{p}^2}{2m} \right| \psi_n \right\rangle = \frac{1}{2}\left\langle \psi_n \,|\, \boldsymbol{r}\cdot\hat{\boldsymbol{r}} \,|\, \psi_n\right\rangle = \frac{1}{2}\left\langle \psi_n \,|\, V(\boldsymbol{r}) \,|\, \psi_n\right\rangle. \tag{3.1.16}$$

Thus, $\frac{2}{3}$ of the total energy (i.e., the rest mass) is in the form of potential energy.

Problem 3.2 (J15Q3)a) Consider the Hamiltonian for a general time-independent, one-dimensional potential $V(x)$,

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x). \tag{3.1.17}$$

Show that for an arbitrary, continuous function $\phi(x)$, the value of

$$E = \frac{\langle\phi|H|\phi\rangle}{\langle\phi|\phi\rangle} \tag{3.1.18}$$

gives an upper bound on the ground state energy for the potential $V(x)$.

b) For a particle moving in a triangular potential well

$$V(x) = \begin{cases} \infty & \text{if } x < 0, \\ V_0 x/L & \text{if } x > 0 \end{cases} \tag{3.1.19}$$

the energy levels take the form

$$E_n = \alpha_n V_0 \left(\frac{\hbar^2}{mL^2V_0}\right)^q \tag{3.1.20}$$

where α_n and q are numerical constants. Determine the value of the exponent q.

c) Using the approach in part (a), find an estimate for the constant α_0 corresponding to the ground state in the triangular potential well. (The estimate needs not be optimal, but should be based on a reasonable variational calculation.)

The variational principle has nothing to do with one dimension or spatial wavefunction. If we decompose into energy eigenstates,

$$|\phi\rangle = \sum a_n |\psi_n\rangle, \tag{3.1.21}$$

with $E_0 < E_1 < \ldots$, then

$$E = \frac{\sum E_n |a_n|^2}{\sum |a_n|^2} \geq E_0. \tag{3.1.22}$$

Since the parameters V_0 and L appear only in the combination $\frac{V_0}{L}$ in the Hamiltonian, the energy levels must be invariant under $V_0 \mapsto \alpha V_0$, $L \mapsto \alpha L$. This gives $q = \frac{1}{3}$.

One possible choice of variational wavefunctions $\phi(x)$ is the family

$$\phi(x;a) = \begin{cases} xe^{-\alpha x} & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases}, \tag{3.1.23}$$

because it vanishes at 0, is normalizable, and leads to easy integrals. We have

$$\langle\phi|H|\phi\rangle = \int_0^\infty \left[\frac{\hbar^2}{2m}\left(-2\alpha x e^{-2\alpha x} + \alpha^2 x^2 e^{-2\alpha x}\right) + \frac{V_0}{L}x^3 e^{-2\alpha x}\right] dx. \tag{3.1.24}$$

Starting from

$$\int_0^\infty e^{-\alpha x}\,dx = \frac{1}{\alpha}, \tag{3.1.25}$$

we can differentiate with respect to α and find

$$\int_0^\infty x^n e^{-\alpha x} = \frac{n!}{\alpha^{n+1}}, \tag{3.1.26}$$

so

$$\langle\phi|H|\phi\rangle = -\frac{\hbar^2}{8m\alpha} + \frac{3V_0}{8\alpha^4 L}. \tag{3.1.27}$$

Similarly, $\langle\phi|\phi\rangle = \frac{1}{4\alpha^2}$. Taking the derivative of E, we find a minimum value of

$$E = \left(\frac{3}{2}\right)^{5/3} V_0 \left(\frac{\hbar^2}{mL^2V_0}\right)^{1/3}, \tag{3.1.28}$$

so $\alpha_0 \leq \left(\frac{3}{2}\right)^{5/3}$.

The spectrum (set of eigenvalues) of the Hamiltonian contains essentially all the physics of a system. In particular, we can restrict attention to the negative eigenvalues, which correspond to bound states of the system. For example, the hydrogen atom, which we will come to later, has an infinite set of bound states with energies arbitrarily close to zero from below.

One of the most important tools for solving a Hamiltonian is symmetry. If there is some operator A which commutes with the Hamiltonian, $[A, H] = 0$, then

$$H(A\,|\psi_n\rangle) = A(H\,|\psi_n\rangle) = E_n(A\,|\psi_n\rangle), \tag{3.1.29}$$

so A respects the eigenspaces of H. This means we can choose a basis for the eigenstates of H such that they are all also eigenstates of A.

Problem 3.3 (M15Q1)
A particle of mass m, is moving on a line under the action of the Hamiltonian:

$$H_L = \frac{p^2}{2m} - b^2\left(\delta(x+L) + \delta(x-L)\right) \tag{3.1.30}$$

(whose potential features a pair of attracting delta functions).

a) State the symmetry, and sketch the shapes of the ground state and of the first excited bound state, in case such a state exists.

b) Determine the minimal L_0 such that for $L > L_0$ the Hamiltonian has more than one bound state.

c) What is the maximal number of bound states that H_L can have?

Clearly there is a symmetry $x \to -x$. This means all the energy eigenstates will be parity eigenstates, i.e., either even or odd functions.

Everywhere except at $\pm L$, the wavefunction has to be an eigenfunction of $\frac{d}{dx^2}$. The eigenfunctions of this operator with eigenvalue k^2 are of the form $e^{\pm kx}$. Typically we take k imaginary to avoid egregious non-normalizability issues, but in this instance we can keep k real by having only an e^{kx} contribution for $x < -L$ and only e^{-kx} for $x > L$. In the middle region we can have a combination of the two. To sketch the ground state and first excited bound state we use the parity symmetry. The two states are shown in Figure 3.2.

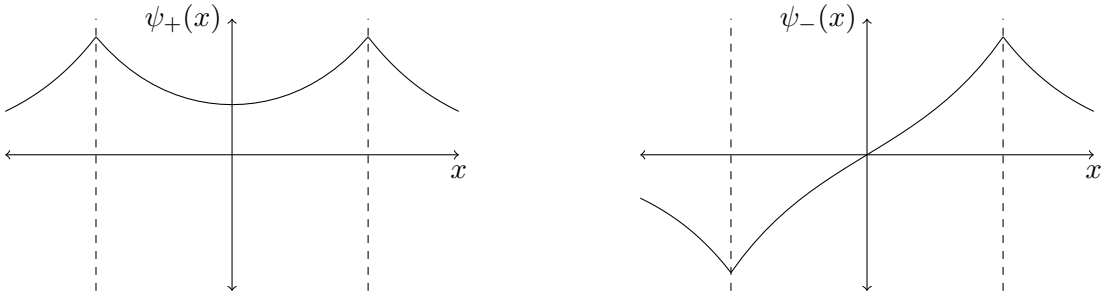


Figure 3.2: The ground state $\psi_+(x)$, with even parity, and the first excited bound state (if it exists) $\psi_-(x)$, with odd parity.

Now for the details. The general form for these two wavefunctions is

$$\psi_\pm(x) = \begin{cases} \pm A e^{k(x+L)} & x < -L \\ \frac{2A}{e^{kL} \pm 1} \left(\pm e^{k(x+L)} + e^{-k(x-L)} \right) & -L < x < L \\ A e^{-k(x-L)} & x > L \end{cases} . \tag{3.1.31}$$

By integrating Schrödinger's equation in a small neighborhood around L or $-L$, we find

$$-\frac{\hbar^2}{2m} \Delta \frac{d\psi}{dx} = b^2 \psi(\pm L), \tag{3.1.32}$$

so

$$\frac{\hbar^2 A k}{2m} \left(1 - \frac{1}{e^{kL} \pm 1} \right) = b^2 A, \tag{3.1.33}$$

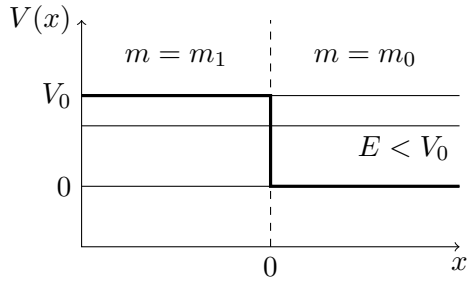
so $k = \frac{2b^2 m}{\hbar^2} \left(1 - \frac{1}{e^{kL} \pm 1} \right)^{-1}$. When we take the positive sign this is always fine, but for the negative sign, we need $e^{kL} > 2$ in order to have a positive k. Thus, $L_0 = \frac{\ln 2}{k}$.

It is clear from everything said so far that there can be only two bound states.

Problem 3.4 (J07Q2)
An electron is moving in one dimension in a potential $V(x) = 0$ for $x > 0$ and $V(x) = V_0 > 0$ for $x < 0$. The region $x > 0$ is empty space, where the electron mass is the usual bare mass m_0, but in the region $x < 0$ it has a modified "effective mass" m_1. When the mass of a non-relativistic particle depends on its position, the Hamiltonian should be written in the operator-ordered form

$$H = \frac{1}{2} p (m(x))^{-1} p + V(x) \tag{3.1.34}$$

where $[x, p] = i\hbar$.



a) The standard continuity conditions (continuity of $\Psi(x)$ and $\Psi'(x) = d\Psi/dx$ only apply at $x = 0$ if $m_1 = m_0$. Derive the continuity conditions that apply at points where the mass is discontinuous.

b) The (unnormalized) wave function of an eigenstate of the Hamiltonian with an energy $E < V_0$ is given by $\Psi(x) = A \sin k(x - x_0)$ for $x > 0$. Find k, x_0 and $\Psi(x)$ for $x < 0$. Make a sketch of the function $\Psi(x)$, indicating its essential features.

Integrating the Schrödinger equation in a neighborhood around $x = 0$ gives

$$\Delta \left(\frac{1}{m(x)} \frac{d\Psi}{dx} \right) = 0, \tag{3.1.35}$$

so in this case $\Psi(x)$ and $\frac{1}{m} \frac{d\Psi}{dx}$ are continuous.

The Schrödinger equation applied to $\Psi(x) = A \sin k(x - x_0)$ immediately gives $E = \frac{\hbar^2 k^2}{2m}$, so $k = \frac{\sqrt{2mE}}{\hbar}$. For $x < 0$, we need $\Psi(x) = Be^{\kappa x}$ with $\kappa = \frac{\sqrt{2m(V_0 - E)}}{\hbar}$. The boundary conditions give

$$B = A \sin(-kx_0), \qquad \frac{B\kappa}{m_1} = \frac{Ak}{m_0} \cos kx_0. \tag{3.1.36}$$

Dividing these equations and solving for x_0, we find

$$x_0 = -\frac{1}{k} \cot^{-1} \frac{m_0 \kappa}{m_1 k}. \tag{3.1.37}$$

The full solution $\Psi(x)$ is sketched in Figure 3.3.

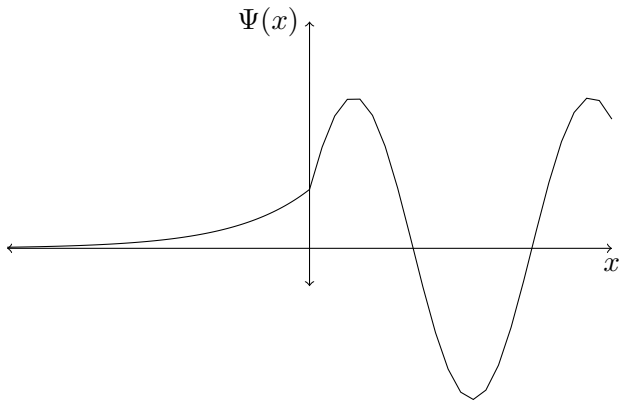


Figure 3.3: The solution $\Psi(x)$, which has a kink at $x = 0$ because of the discontinuity in effective mass.

A cute result which rarely comes up: in one dimension, the nth energy eigenstate has $n - 1$ nodes. This is a consequence of fancy Sturm-Liouville business, but there's a pretty short way to prove it. Start with a potential $V(x)$, and let $V_\epsilon(x)$ be $V(x)$ in $(-\epsilon, \epsilon)$ and infinite elsewhere. For very small ϵ, this looks like the particle-in-a-box, for which we can explicitly write down the eigenstates and show that they obey the node law which we claim. If we then increase ϵ, the node law is preserved. Indeed, for a state to develop a node, both ψ and its derivative would have to vanish somewhere, and this would force ψ to vanish everywhere.

Problem 3.5 (M06Q1)
A particle of mass m moves in one dimension to the right of a wall at $x = 0$ in the potential

$$V(x) = -\frac{A}{x} \tag{3.1.38}$$

where A is a given positive parameter.

a) Find the ground state energy.

b) Find the position expectation value, $\langle x \rangle$, for the ground state.

We need a function which vanishes at $x = 0$, which has no nodes (so it can be the ground state), and which is integrable over $(0, \infty)$. An obvious choice is $\psi(x) \propto xe^{-\alpha x}$. Indeed, this gives

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} - \frac{A}{x}\psi = \frac{\alpha\hbar^2/m - A}{x} - \frac{\hbar^2\alpha^2}{2m}. \tag{3.1.39}$$

Choosing $\alpha = \frac{Am}{\hbar^2}$, we obtain a state with energy $E_0 = -\frac{A^2m}{2\hbar^2}$.

The expectation value of position in the ground state is

$$\langle x \rangle = \frac{\langle\psi|x|\psi\rangle}{\langle\psi|\psi\rangle} = \frac{3}{2\alpha} = \frac{3\hbar^2}{2Am} = -\frac{3A}{4E_0}. \tag{3.1.40}$$

3.2 Harmonic Oscillators

The Hamiltonian for a harmonic oscillator is

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2. \tag{3.2.1}$$

Figure 3.4

If we introduce the operators

$$a = \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{ip}{m\omega}\right), \qquad a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{ip}{m\omega}\right), \tag{3.2.2}$$

then we find

$$H = \hbar\omega\left(a^\dagger a + \frac{i}{2\hbar}[p,x]\right) = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right). \tag{3.2.3}$$

These operators satisfy $[a, a^\dagger] = 1$, which implies $[H, a^\dagger] = \hbar\omega a^\dagger$ and $[H, a] = -\hbar\omega a$. Let $H|\psi\rangle = E|\psi\rangle$; then we have

$$H(a|\psi\rangle) = aH|\psi\rangle - \hbar\omega a|\psi\rangle = (E - \hbar\omega)a|\psi\rangle, \tag{3.2.4}$$

so $a|\psi\rangle$ is a state with energy $E - \hbar\omega$. Similarly, $a^\dagger|\psi\rangle$ has energy $E + \hbar\omega$. Clearly the Hamiltonian is positive definite, so the energy levels must be bounded from below, meaning we can't keep acting with a indefinitely; the only way for the energies to stop falling is to have a state $|0\rangle$ for which

$$a|0\rangle = 0. \tag{3.2.5}$$

This is the ground state of the Hamiltonian, and clearly it has energy $\frac{1}{2}\hbar\omega$. By acting with $a^\dagger$, we can construct all other states,

$$|n\rangle \propto (a^\dagger)^n|0\rangle, \tag{3.2.6}$$

and $|n\rangle$ has energy $\hbar\omega\left(n + \frac{1}{2}\right)$. To obtain the exact proportionality, we note that

$$\left\langle n \middle| aa^\dagger \middle| n \right\rangle = \left\langle n \middle| \left(a^\dagger a + 1\right) \middle| n \right\rangle = n + 1, \tag{3.2.7}$$

so

$$a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle. \tag{3.2.8}$$

This was really easy, so it's pretty convenient that basically everything is a harmonic oscillator.

Problem 3.6 (M08Q2)
The dynamics of a system is characterized by the Hamiltonian

$$H = a^\dagger a + \frac{1}{2}, \qquad [a, a^\dagger] = 1. \tag{3.2.9}$$

a) Show that the ground state of this system, $|0\rangle$ satisfies

$$a\,|0\rangle = 0. \tag{3.2.10}$$

b) Consider the state

$$|\alpha\rangle = \mathcal{N} e^{\alpha a^\dagger - \alpha^* a}\,|0\rangle \tag{3.2.11}$$

where $\mathcal{N}$ is some normalization constant. Show that $a\,|\alpha\rangle = \alpha\,|\alpha\rangle$. Find $\mathcal{N}$.

c) Consider the change of variables

$$a = \frac{1}{\sqrt{2}}\,(q + ip)\,, \qquad a^\dagger = \frac{1}{\sqrt{2}}\,(q - ip)\,. \tag{3.2.12}$$

Derive and interpret the hamiltonian in this set of new variables.

d) Calculate $\langle\alpha|q|\alpha\rangle$. Describe the time dependence of $\langle\alpha|q|\alpha\rangle$.

Oops, we already showed that $a\,|0\rangle = 0$. It's because the Hamiltonian has to have energy levels bounded from below.

Using the commutation relations of a and $a^\dagger$, we have

$$a\left(\alpha a^\dagger - \alpha^* a\right)^n = \alpha n \left(\alpha a^\dagger - \alpha^* a\right)^{n-1} + \left(\alpha a^\dagger - \alpha^* a\right)^n a. \tag{3.2.13}$$

The second term annihilates $|0\rangle$, so we have

$$a\,|\alpha\rangle = \mathcal{N} a \left(\sum_{n=0}^{\infty} \frac{1}{n!}\left(\alpha a^\dagger - \alpha^* a\right)^n\right)|0\rangle = \mathcal{N}\alpha \left(\sum_{n=0}^{\infty} \frac{1}{(n-1)!}\left(\alpha a^\dagger - \alpha^* a\right)^{n-1}\right)|0\rangle = \alpha\,|\alpha\rangle\,. \tag{3.2.14}$$

To determine $\mathcal{N}$, we use the Baker-Campbell-Hausdorff formula to show that

$$e^{\alpha a^\dagger} e^{-\alpha^* a} = e^{\alpha a^\dagger - \alpha^* a + \frac{1}{2}|\alpha|^2}. \tag{3.2.15}$$

We then have

$$\langle\alpha|\alpha\rangle = \mathcal{N}^2 \left\langle 0 \left| e^{\alpha^* a - \alpha a^\dagger} \right| \alpha \right\rangle = \mathcal{N}^2 e^{-|\alpha|^2/2}\,\langle 0| e^{\alpha a^\dagger} e^{-\alpha^* a}|0\rangle = \mathcal{N}^2 e^{|\alpha|^2/2}, \tag{3.2.16}$$

so $\mathcal{N} = e^{|\alpha|^2/4}$.

Using the given change of variables, we have

$$[a, a^\dagger] = i[p, q], \tag{3.2.17}$$

so $[q, p] = i$. The Hamiltonian is then

$$H = \frac{1}{2}(p^2 + q^2) + \frac{i}{2}[p, q] + \frac{1}{2} = \frac{1}{2}(p^2 + q^2), \tag{3.2.18}$$

which describes a harmonic oscillator.

The expectation of position in the state $|\alpha\rangle$ is

$$\langle\alpha|q|\alpha\rangle = \left\langle \alpha \left| \frac{a+a^\dagger}{2} \right| \alpha \right\rangle = \operatorname{Re}\alpha. \tag{3.2.19}$$

The time dependence is most easily computed using the Heisenberg picture. We have

$$\frac{d}{dt}\langle\alpha|q|\alpha\rangle = \frac{1}{i}\langle\alpha\,|\,[q,H]\,|\,\alpha\rangle = \left\langle \alpha \left| \frac{a-a^\dagger}{2} \right| \alpha \right\rangle = \operatorname{Im}\alpha, \tag{3.2.20}$$

so it only remains to determine the behavior of α itself. We have

$$\frac{d}{dt}\langle\alpha|a|\alpha\rangle = \frac{1}{i}\langle\alpha\,|\,[a,H]\,|\,\alpha\rangle = -i\alpha, \tag{3.2.21}$$

so $\alpha \sim e^{-it}$, and hence $\langle\alpha|q|\alpha\rangle \sim \cos t$.

Problem 3.7 (J06Q1)a) For a particle moving in three dimensions with Hamiltonian

$$H = \frac{p^2}{2m} + V(\boldsymbol{r}) \tag{3.2.22}$$

in an arbitrary quantum state, what are the time derivatives, $d\langle\boldsymbol{r}\rangle/dt$ and $d\langle\boldsymbol{p}\rangle/dt$, of the expectation values of the position and momentum?

b) For times $t < 0$, a one-dimensional simple harmonic oscillator with mass m and frequency ω is in its ground state, at energy $\hbar\omega/2$, with $\langle x\rangle = 0$. At time $t = 0$ a uniform electric field E is instantaneously turned on and remains on for $t > 0$; it couples to the particle's charge q. What is the full time- and x-dependence of the particle's wave function $\psi(x,t)$? If its energy is measured at time t, what are the possible results and their probabilities?

Working in the Heisenberg picture, we immediately find

$$\frac{d\langle\boldsymbol{r}\rangle}{dt} = \left\langle \psi \left| \frac{p}{m} \right| \psi \right\rangle, \qquad \frac{d\langle\boldsymbol{p}\rangle}{dt} = -\langle\psi\,|\,\nabla V(\boldsymbol{r})\,|\,\psi\rangle\,. \tag{3.2.23}$$

When the electric field turns on, the Hamiltonian becomes

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2 - qEx = \frac{p^2}{2m} + \frac{1}{2}m\omega^2\left(x - \frac{qE}{m\omega^2}\right)^2 + \text{const}, \tag{3.2.24}$$

so the relevant position operator is now $x_E = x - \frac{qE}{m\omega^2} \equiv x - x_0$. The new ladder operators are

$$a_E = \sqrt{\frac{m\omega}{2\hbar}}\left(x_E + \frac{ip}{m\omega}\right) = a - \frac{qE}{\omega\sqrt{2\hbar m\omega}}, \qquad a_E^\dagger = a^\dagger - \frac{qE}{\omega\sqrt{2\hbar m\omega}}. \tag{3.2.25}$$

Before the electric field was turned on, the system was in the state $|0\rangle$ with $a|0\rangle = 0$, so

$$a_E|0\rangle = -\sqrt{\frac{m\omega}{2\hbar}}x_0|0\rangle, \tag{3.2.26}$$

Eigenstates of the lowering operator are coherent states (as described in the previous problem), so the time-evolving state is given by

$$\left|-\sqrt{\frac{m\omega}{2\hbar}}x_0 e^{-i\omega t}\right\rangle, \tag{3.2.27}$$

where $|\alpha\rangle \propto e^{\alpha a_E^\dagger - \alpha^* a_E}|0_E\rangle$. Now we need to write this in position space. We start by computing

$$\alpha a_E^\dagger - \alpha^* a_E = i\sqrt{\frac{2m\omega}{\hbar}}\left((\operatorname{Im}\alpha)x_E - (\operatorname{Re}\alpha)\frac{p}{m\omega}\right). \tag{3.2.28}$$

From Baker-Campbell-Hausdorff we have

$$e^{x_E}e^{p} = e^{x_E+p+i\hbar/2}, \tag{3.2.29}$$

so

$$\exp\left(i\sqrt{\frac{2m\omega}{\hbar}}\left((\operatorname{Im}\alpha)x_E - (\operatorname{Re}\alpha)\frac{p}{m\omega}\right)\right) \tag{3.2.30}$$

$$= \exp\left(i\sqrt{\frac{2m\omega}{\hbar}}(\operatorname{Im}\alpha)x_E\right)\exp\left(-i\sqrt{\frac{2}{\hbar m\omega}}(\operatorname{Re}\alpha)p\right)\exp\left(-i\frac{\alpha^2-(\alpha^*)^2}{4}\right). \tag{3.2.31}$$

Now we act on the ground state $\langle x|0_E\rangle = \psi_0(x-x_0)$, where $\psi_0(x) = \langle x|0\rangle$ is the well-known ground state of the harmonic oscillator. The second factor above is a translation operator. Substituting $\alpha = -\sqrt{\frac{m\omega}{2\hbar}}x_0 e^{-i\omega t}$, the result is

$$\psi(x,t) \propto \exp\left(i\frac{m\omega}{\hbar}x_0(x-x_0)\sin\omega t\right)\exp\left(-\frac{m\omega}{2\hbar}x_0^2\sin(2\omega t)\right)\psi_0\left(x - x_0 + x_0\cos(\omega t)\right), \tag{3.2.32}$$

a wavefunction only a mother could love. We still need to normalize this, since we never normalized the coherent state, but this is easy since we already know $\psi_0(x)$ is normalized. We can just drop the second factor, and so we have

$$\psi(x,t) = \exp\left(i\frac{m\omega}{\hbar}x_0(x-x_0)\sin\omega t\right)\psi_0\left(x - x_0 + x_0\cos(\omega t)\right). \tag{3.2.33}$$

Now its father loves it too.

Since coherent states are eigenvalues of the lowering operator, and $a|n\rangle = \sqrt{n}|n-1\rangle$, we must have

$$|\alpha\rangle = e^{-|\alpha|^2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle. \tag{3.2.34}$$

Thus, the probability of measuring energy $E_n = \hbar\omega\left(n+\frac{1}{2}\right)$ at time $t>0$ is

$$P_n(t) = \frac{|\alpha|^{2n}e^{-|\alpha|^2}}{n!}, \tag{3.2.35}$$

the familiar Poisson distribution with $\langle n\rangle = |\alpha|^2 = \frac{m\omega}{2\hbar}x_0^2$, or equivalently,

$$\langle E\rangle = \frac{1}{2}m\omega^2 x_0^2. \tag{3.2.36}$$

This is to be expected, since this is obviously the expected energy of the wavefunction at $t=0$.

Problem 3.8 (M02Q1)
Consider the driven harmonic oscillator:

$$H(t) = \frac{1}{2}\left(p^2 + x^2\right) - \sqrt{2}f(t)x \tag{3.2.37}$$

where $f(t)$ is a c-number function of time. Note that we have set the natural frequency of the oscillator, ω_0, to 1. In the following you should also use $\hbar = 1$.

Defining the time evolution operator in the Schrödinger picture by,

$$|\psi(t)\rangle_S = U(t)\,|\psi(0)\rangle\,, \tag{3.2.38}$$

we can transform to the Heisenberg picture

$$|\psi\rangle_H = U^\dagger(t)\,|\psi(t)\rangle_S \equiv |\psi(0)\rangle \tag{3.2.39}$$

and

$$O_H(t) = U^\dagger(t) O_S U(t). \tag{3.2.40}$$

Working in the Heisenberg picture,

a) Write down the equations of motion for the operators x and p.

b) Solve the operator equations of motion derived in a) for the case

$$f(t) = \begin{cases} f_0 \cos\omega t & \text{for } 0 \le t \le T, \\ 0 & \text{otherwise} \end{cases}. \tag{3.2.41}$$

c) Compute the expectation value of the total energy gained by the oscillator at resonance, $\omega = 1$, if $|\psi(0)\rangle = |0\rangle$, its unperturbed ground state. Sketch your result as a function of T.

The equations of motion for the operators are

$$\frac{dx}{dt} = p, \qquad \frac{dp}{dt} = -x + \sqrt{2}f(t). \tag{3.2.42}$$

To solve these equations for the given $f(t)$, we take another derivative and use

$$\frac{d^2p}{dt^2} + p = \sqrt{2}\dot{f}(t). \tag{3.2.43}$$

A Green's function for this equation is $p(t) = \sin t$, so

$$p(t) = -\sqrt{2}\omega \int_0^t \sin\left(T - t\right) \sin\left(\omega t\right)\, dt. \tag{3.2.44}$$

This then determines

$$x(t) = \int_0^t p(t)\, dt. \tag{3.2.45}$$

One could evaluate these integrals in the general case, if one hated oneself.

For $\omega = 1$ things simplify considerably. Using

$$\sin(T-t)\sin(t) = \frac{1}{2}\left(\cos(T-2t) - \cos T\right), \tag{3.2.46}$$

we find

$$p(t) = \frac{1}{\sqrt{2}}\left(t\cos T + \frac{1}{2}\left(\sin(T) - \sin(T-2t)\right)\right). \tag{3.2.47}$$

Another integral gives

$$x(t) = \frac{1}{4\sqrt{2}}\left((2t^2-1)\cos T - 2t\sin T + \cos(T-2T)\right). \tag{3.2.48}$$

The expected energy gain is

$$\langle \Delta E\rangle = \frac{1}{2}\left(x(T)^2 + p(T)^2\right) = \frac{4+T^2}{16}\left(\sin T - T\cos T\right)^2. \tag{3.2.49}$$

Harmonic oscillators are so easy because they have quadratic Hamiltonians. This implies in particular that when we look at a harmonic oscillator in more than one dimension, it can be viewed as a sum of harmonic oscillators in each direction. The energy levels will be degenerate, since there are many ways of forming a fixed sum from the d ladder levels.

Problem 3.9 (J15Q1)
Consider a toy model of the Helium atom where the Coulombic interaction potential is replaced with a Hooke's law potential. If the nucleus of the atom is located at $\boldsymbol{r} = 0$ and the electrons of mass m have position vectors $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$, the interaction potential is

$$V(\boldsymbol{r}_1, \boldsymbol{r}_2) = \frac{1}{2}m\omega^2\left(\boldsymbol{r}_1^2 + \boldsymbol{r}_2^2\right) - \frac{\lambda}{4}m\omega^2\left(\boldsymbol{r}_1 - \boldsymbol{r}_2\right)^2. \tag{3.2.50}$$

This model is exactly solvable. Assume $\lambda > 0$.

a) What constraint must be imposed on λ for the system to be well-behaved?

b) What are the energy levels of this system when $\lambda = 1/2$?

c) Taking into account the spin of the electrons, what are the degeneracies of the lowest four energy levels when $\lambda = 1/2$?

d) Suppose the Helium atom is initially in the third excited state. It then undergoes a decay through an electric dipole transition to a lower-energy state. What are the possible energies of the emitted photon?

In terms of the center of mass $\boldsymbol{u} = \frac{1}{2}(\boldsymbol{r}_1 + \boldsymbol{r}_2)$ and the separation $\boldsymbol{v} = \boldsymbol{r}_1 - \boldsymbol{r}_2$, the interaction potential is

$$V = m\omega^2 \boldsymbol{u}^2 + \frac{1-\lambda}{4} m\omega^2 \boldsymbol{v}^2. \tag{3.2.51}$$

Thus, in order for the potential to be positive definite, we need $\lambda < 1$.

The full Hamiltonian when $\lambda = 1/2$ is

$$H = \frac{\boldsymbol{p}_u^2/2 + 2\boldsymbol{p}_v^2}{2m} + V = \left(\frac{\boldsymbol{p}_u^2}{2(2m)} + \frac{1}{2}(2m)\omega^2 \boldsymbol{u}^2\right) + \left(\frac{\boldsymbol{p}_v^2}{2(m/2)} + \frac{1}{2}\left(\frac{m}{2}\right)\left(\frac{\omega}{\sqrt{2}}\right)^2 \boldsymbol{v}^2\right). \tag{3.2.52}$$

We have an oscillator with frequency ω describing the excitations in $\boldsymbol{u}$, and an oscillator with frequency $\frac{1}{\sqrt{2}}\omega$ describing the excitations in $\boldsymbol{v}$. Both of these oscillators are in three dimensions, so we have six quantum numbers to label the energy levels:

$$E_{a_1,a_2,a_3,b_1,b_2,b_3} = \hbar\omega\left(a_1 + a_2 + a_3 + \frac{3}{2}\right) + \hbar\frac{\omega}{\sqrt{2}}\left(b_1 + b_2 + b_3 + \frac{3}{2}\right). \tag{3.2.53}$$

To find the degeneracy of the energy levels, we need to count the number of ways the sums $a_1 + a_2 + a_3$ and $b_1 + b_2 + b_3$ can be arranged, and also count the spin degeneracy. The total spin is set by the need to obey Fermi-Dirac statistics. Exactly one of the spatial and spin wavefunctions should be antisymmetric. The antisymmetric spin state is the singlet, and the symmetric states are in the triplet. The symmetry of the spatial wavefunction is given by the parity of $b_1 + b_2 + b_3$. So, we have the following degeneracies.

Level	$a_1 + a_2 + a_3$	$b_1 + b_2 + b_3$	$(E - E_0)/\hbar\omega$	Spin State	Degeneracy
0	0	0	0	Singlet	1×1
1	0	1	$\frac{1}{\sqrt{2}}$	Triplet	3×3
2	1	0	1	Singlet	3×1
3	0	2	$\sqrt{2}$	Singlet	6×1

Spoilers for perturbation theory: the dipole selection rule comes from determining whether the matrix element $\langle\psi_f|\mathcal{O}|\psi_i\rangle$ vanishes, where $\mathcal{O}$ is the dipole moment operator which transforms as a vector, i.e., odd under inversions. Clearly ψ_i and ψ_f have to have different parities in order for this matrix element to be nonzero. The only lower energy state with parity different from the third excited state is the first excited state, so the photons will have energy $\frac{\hbar\omega}{\sqrt{2}}$.

Problem 3.10 (M15Q2)
A point particle of mass m and electric charge q moves in a 3d harmonic oscillator potential with frequency ω and a uniform electric field of strength $\mathcal{E}$ pointing in the z-direction. The Hamiltonian is

$$H = \frac{\boldsymbol{p}^2}{2m} + \frac{1}{2}m\omega^2 \boldsymbol{r}^2 - q\mathcal{E}\hat{\boldsymbol{z}} \tag{3.2.54}$$

a) What are the eigenenergies of this Hamiltonian?

b) Find an expression for the ground state wave-function.

Assume now that the system is described by the above Hamiltonian only for $t < 0$, and that at $t = 0$ the electric field is suddenly turned off. For $t < 0$, the system is in its ground state.

c) What is the probability that the system will end up in the new ground state right after the electric field is turned off?

d) What is the expectation value of the electric dipole moment $\boldsymbol{d} = q\boldsymbol{r}$ at some given time $t > 0$?

This is almost identical to a problem we did before, so a bit less detail here. This Hamiltonian describes a harmonic oscillator with frequency ω and equilibrium position $z_0\hat{\boldsymbol{z}}$, where $z_0 = \frac{q\mathcal{E}}{m\omega^2}$. Its eigenenergies are $E_n = \hbar\omega\left(n + \frac{3}{2}\right)$.

The ground state wavefunction is $\psi_0(\boldsymbol{r} - z_0\boldsymbol{z})$, where ψ_0 is the ground state wavefunction for the unperturbed harmonic oscillator.

Since the shifted ground state is an eigenstate of the unshifted lowering operator, it is a coherent state of the unshifted Hamiltonian with $\alpha = \sqrt{\frac{m\omega}{2\hbar}}z_0$. The probability of ending up in the new ground state is

$$P_0 = e^{-|\alpha|^2/2} = \exp\left(-\frac{q^2\mathcal{E}^2}{4m\hbar\omega^3}\right). \tag{3.2.55}$$

By axisymmetry, the expectation of the dipole moment will point along the z-axis for all times. We know that $\langle z\rangle = \operatorname{Re}\alpha$ and that $\alpha(t) = \alpha(0)e^{-i\omega t}$, so $\langle\boldsymbol{d}\rangle = qz_0\cos(\omega t)$.

3.3 Spin and Angular Momentum

In Chapter 1, we used the conservation of angular momentum to simplify problems with rotational symmetry. The same approach works in quantum mechanics, under a different guise.

In the Heisenberg picture, we showed that

$$i\hbar\frac{dA}{dt} = [A, H] + i\hbar\frac{\partial A}{\partial t}. \tag{3.3.1}$$

Thus, if an operator has no explicit time dependence, then its conservation is equivalent to it commuting with the Hamiltonian.

If the Hamiltonian is invariant under rotational symmetry, then we must have

$$H = R(\theta, \hat{\boldsymbol{n}})^{-1} H R(\theta, \hat{\boldsymbol{n}}), \tag{3.3.2}$$

where $R(\theta, \hat{\boldsymbol{n}})$ is a rotation operator. These operators live in the group $SO(3)$, which is three-dimensional. They can be expressed as exponentials of elements of the algebra $\mathfrak{so}(3)$, via

$$R(\theta, \hat{\boldsymbol{n}}) = \exp\left(\frac{i}{\hbar}\theta\hat{\boldsymbol{n}}\cdot\boldsymbol{J}\right) = 1 + \frac{i}{\hbar}\theta\hat{\boldsymbol{n}}\cdot\boldsymbol{J} + \mathcal{O}(\theta^2). \tag{3.3.3}$$

Thus, at first order in θ, the relation above reduces to

$$[H, \boldsymbol{J}] = 0, \tag{3.3.4}$$

meaning that the operators $\boldsymbol{J}$ are conserved.

We noted previously that if an operator commutes with the Hamiltonian, we can choose the energy eigenstates such that they are also eigenstates of the conserved operator. Now we have three operators which each commute with the Hamiltonian (J_x, J_y, and J_z). However, they do not commute with each other; the $\mathfrak{so}(3)$ algebra (accounting for the factor of $\hbar$ we have inserted by hand) is

$$[J_i, J_j] = i\hbar\epsilon_{ijk}J_k. \tag{3.3.5}$$

Thus, we can only choose the energy eigenstates to be eigenstates of one of the $\boldsymbol{J}$ operators. In addition, it is straightforward to show that

$$[J^2, J_i] = 0, \tag{3.3.6}$$

and clearly the Hamiltonian commutes with $J^2 = J_x^2 + J_y^2 + J_z^2$. So in summary, we can choose energy eigenstates which are also eigenstates of J^2 and J_z.

We can construct the eigenstates of J^2 and J_z using an approach similar to the ladder operators of the harmonic oscillator. Note that

$$[J_z, J_x \pm iJ_y] = \pm\hbar(J_x \pm iJ_y), \tag{3.3.7}$$

so the operators $J_\pm \equiv J_x \pm iJ_y$ raise and lower the eigenvalue of J_z in units of $\hbar$. To determine the exact relationship, let $|j, m\rangle$ be a simultaneous eigenstate of J^2 and J_z, with

$$J^2 |j, m\rangle = \hbar^2 j(j+1) |j, m\rangle , \qquad J_z |j, m\rangle = m\hbar |j, m\rangle . \tag{3.3.8}$$

We have not yet shown that j is a half-integer, so the first relation is just foreshadowing. We then have

$$\begin{aligned} \langle j, m | J_- J_+ | j, m\rangle &= \left\langle j, m \,\middle|\, \left(J_x^2 + J_y^2 + i[J_x, J_y]\right) \,\middle|\, j, m\right\rangle && (3.3.9)\\ &= \left\langle j, m \,\middle|\, \left(J^2 - J_z^2 - J_z\right) \,\middle|\, j, m\right\rangle && (3.3.10)\\ &= \hbar^2 \left(j(j+1) - m(m+1)\right) |j, m\rangle . && (3.3.11) \end{aligned}$$

It follows that

$$J_\pm |j, m\rangle = \hbar\sqrt{j(j+1) - m(m\pm1)}\, |j, m\pm1\rangle . \tag{3.3.12}$$

We should not be able to raise and lower the eigenvalue of J_z indefinitely, since $J_z^2 < J^2$. This means there must be a state $|j, m_+\rangle$ with $J_+ |j, m_+\rangle = 0$. This is called the highest-weight state in mathematical literature, and from the above it satisfies either $m_+ = j$ or $m_+ = -j-1$. The second possibility would violate $J_z^2 < J^2$, so $m_+ = j$. Similarly, there must be a lowest-weight state $|j, m_-\rangle$ such that $J_- |j, m_-\rangle = 0$, and the relations above imply $m_- = -j$. In order to avoid an infinite descent starting from $|j, j\rangle$, we must eventually reach $|j, -j\rangle$, which implies $j - (-j) = 2j \in \mathbb{Z}$. Hence, j is a half-integer.

We now have all the information we need about states of fixed J^2 and J_z. They are organized into multiplets labeled by $j \in \frac{1}{2}\mathbb{Z}$, which contain $2j+1$ states $|j, j\rangle$ through $|j, -j\rangle$.

Figure 3.5: Spin represents intrinsic angular momentum.

Everything said so far applies to the total angular momentum of a system. An additional fact, which is hard to motivate well without looking at its relativistic origin, is that particles can carry internal angular momentum in the form of spin. The total angular momentum then splits as

$$\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S}, \tag{3.3.13}$$

where $\boldsymbol{L}$ is orbital angular momentum and $\boldsymbol{S}$ is the spin piece.

Spin represents intrinsic magnetic moment in a particle, so it couples to magnetic fields. In the spin-1/2 case, this coupling is represented by

$$H = \mu_B \boldsymbol{B} \cdot \boldsymbol{\sigma} \tag{3.3.14}$$

where

$$\boldsymbol{\sigma} = \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\} \tag{3.3.15}$$

are the Pauli matrices which form a representation of the $\mathfrak{so}(3)$ algebra:

$$[\sigma_i, \sigma_j] = 2\epsilon_{ijk}\sigma_k, \tag{3.3.16}$$

and $\mu_B = \frac{e\hbar}{2mc}$ is the Bohr magneton (or a similar quantity if the particle is not an electron).

Problem 3.11 (M08Q3)
A spin of $s = \frac{1}{2}$ has its z-component "up" at time $t = 0$. The dynamics of the spin are given by the Hamiltonian

$$H = \lambda\hbar\sigma_x, \tag{3.3.17}$$

where σ_x is the usual Pauli matrix for a spin-1/2.

a) If the z-component of this spin is measured at time $t = \tau$, what are the probabilities of each possible result of this measurement?

b) Now consider a slightly different question: the spin again starts "up" at time $t = 0$, but now its z-component is measured twice, once at time $t = \tau/2$ and then again at time $t = \tau$. The above Hamiltonian gives the spin's dynamics between the measurements, and you can assume the measurements happen instantaneously. However, the result of the first measurement at time $t = \tau/2$ is not known to you. Now what are the probabilities of each possible result of the second measurement at time $t = \tau$?

We are only considering the spin degree of freedom for this system, and so the state space is $2s + 1 = 2$-dimensional. We can express the state as

$$|\psi(t)\rangle = \alpha(t)\,|+\rangle + \beta(t)\,|-\rangle\,, \tag{3.3.18}$$

where $|\pm\rangle = \left|\frac{1}{2}, \pm\,|1\rangle\,2\right\rangle$ and $\alpha(0) = 1$, $\beta(0) = 0$. The Schrödinger equation then reads

$$\begin{pmatrix}\dot{\alpha}(t)\\ \dot{\beta}(t)\end{pmatrix} = -i\lambda \begin{pmatrix}\beta(t)\\ \alpha(t)\end{pmatrix}. \tag{3.3.19}$$

Taking another time derivative gives $\ddot{\alpha}(t) = -\lambda^2\alpha(t)$ and likewise for $\beta(t)$, so we solve and find

$$|\psi(t)\rangle = \cos(\lambda t)\,|+\rangle + \sin(\lambda t)\,|-\rangle\,. \tag{3.3.20}$$

Thus, the probability of measuring $J_z = \frac{\hbar}{2}$ at time τ is $\cos^2(\lambda\tau)$, and the probability of measuring $-\frac{\hbar}{2}$ is $\sin^2(\lambda\tau)$.

If we make the intermediate measurement at time $\tau/2$, then a simple analysis of the two possibilities shows that the probabilities become

$$P_{\hbar/2} = \cos^4\left(\lambda\frac{\tau}{2}\right) + \sin^4\left(\lambda\frac{\tau}{2}\right), \tag{3.3.21}$$

$$P_{-\hbar/2} = 2\sin^2\left(\lambda\frac{\tau}{2}\right)\cos^2\left(\lambda\frac{\tau}{2}\right). \tag{3.3.22}$$

Note that for small τ, the probability of measuring a spin-flip is unchanged by the intermediate measurement.

Problem 3.12 (M13Q1)
The hyperfine structure of the $n = 1$ level of the hydrogen atom arises from a coupling between the electron spin $\boldsymbol{S}_e$ and the proton spin $\boldsymbol{S}_p$ with Hamiltonian

$$H = A\boldsymbol{S}_e \cdot \boldsymbol{S}_p, \tag{3.3.23}$$

where A is a positive constant. Use the convention where the spin operators are dimensionless. The kinetic energy and Coulomb interaction do not lift the spin degeneracies and may be ignored in this problem.

a) What are the energies and degeneracies of the hyperfine levels in the absence of a magnetic field? A uniform magnetic field $\boldsymbol{B}$ is turned on for a period of time. Assume that the field is constant for $0 < t < T$ and zero for all other times.

b) To a good approximation you can ignore the coupling of the proton spin to the magnetic field, compared to that of the electron spin. Briefly explain why this is true.

c) Assume the atom was in the hyperfine state with total spin zero for $t < 0$. What is the probability that it remains in this state for $t > T$?

The spin coupling can be written as

$$H = \frac{A}{2}\left((\boldsymbol{S}_e + \boldsymbol{S}_p)^2 - S_e^2 - S_p^2\right). \tag{3.3.24}$$

This is useful, since $S_e^2 = S_p^2 = \frac{3}{4}\hbar^2$. It only remains to determine $(\boldsymbol{S}_e + \boldsymbol{S}_p)^2$. Certainly if both the electron and proton spin are aligned upwards then the total spin is 1. Acting with the lowering operator $J_- = J_- \otimes 1 + 1 \otimes J_-$, we find a spin-1 triplet of states with $(\boldsymbol{S}_e + \boldsymbol{S}_p)^2 = 2\hbar^2$. The remaining state is a spin-0 singlet with $(\boldsymbol{S}_e + \boldsymbol{S}_p)^2 = 0$. Thus, the energies are $\frac{A}{4}\hbar^2$ with degeneracy 3 and $-\frac{3A}{4}\hbar^2$ with degeneracy 1.

The coupling of spin to a magnetic field occurs through the magnetic dipole moment. The magnetic dipole moment of a charged body is proportional to its angular momentum, but the constant scaling between them will be inversely proportional to the mass of the body. Since the proton is about 2000 times heavier than the electron, its coupling to the magnetic field is negligible in comparison to that of the electron.

We are given that the atom is in the single spin-zero state for $t < 0$. By explicitly working out the spin-1 triplet above, we find that the unique state orthogonal to it is

$$|\psi(t<0)\rangle = \frac{1}{\sqrt{2}}\left(|+-\rangle - |-+\rangle\right), \tag{3.3.25}$$

where the first spin is the electron. Without loss of generality assume the applied $\boldsymbol{B}$ field is in the z-direction. Then both states $|+-\rangle$ and $|-+\rangle$ are eigenstates of the $\mu_B \boldsymbol{B}\cdot\boldsymbol{\sigma}$ part of the Hamiltonian, so the state for $t > 0$ will be a combination of the spin-zero state and the spin-one state with $J_z = 0$. That is,

$$|\psi(0<t<T)\rangle = \frac{\alpha(t)}{\sqrt{2}}\left(|+-\rangle - |-+\rangle\right) + \frac{\beta(t)}{\sqrt{2}}\left(|+-\rangle + |-+\rangle\right). \tag{3.3.26}$$

The Schrödinger equation gives

$$\begin{pmatrix}\dot\alpha(t)\\ \dot\beta(t)\end{pmatrix} = -\frac{i}{4\hbar}\begin{pmatrix}A\hbar & 2\mu_B B\\ 2\mu_B B & -3A\hbar\end{pmatrix}\begin{pmatrix}\alpha(t)\\ \beta(t)\end{pmatrix}. \tag{3.3.27}$$

The eigenvalues and eigenvectors of the matrix are

$$\lambda_\pm = \frac{i}{4}\left(-A \pm \frac{2}{\hbar}\sqrt{\mu_B^2 B^2 + A^2\hbar^2}\right), \qquad v_\pm = \begin{pmatrix}\frac{A\hbar}{\mu_B B} \pm \frac{1}{\mu_B B}\sqrt{\mu_B^2 B^2 + A^2\hbar^2}\\ 1\end{pmatrix}. \tag{3.3.28}$$

The initial state is $\frac{\mu_B B}{2\sqrt{\mu_B^2 B^2 + A^2\hbar^2}}(v_+ - v_-)$, so at time T we have

$$\alpha(T) = \exp\left(-\frac{iTA}{4}\right)\left(\cos\frac{T\sqrt{\mu_B^2 B^2 + A^2\hbar^2}}{2\mu_B B} + \frac{iA\hbar}{\sqrt{\mu_B^2 B^2 + A^2\hbar^2}}\sin\frac{T\sqrt{\mu_B^2 B^2 + A^2\hbar^2}}{2\mu_B B}\right). \tag{3.3.29}$$

Or something like that, it's a lot of constants to keep track of. The probability of finding the system in the spin-zero state is $|\alpha(T)|^2$. We can ignore the overall phase factor in $\alpha(T)$, and the cross terms cancel, and we can rearrange trig functions, so we have

$$|\alpha(T)|^2 = 1 - \frac{1}{1+\frac{A^2\hbar^2}{\mu_B^2 B^2}} \sin^2\left(\frac{T}{2}\sqrt{1+\frac{A^2\hbar^2}{\mu_B^2 B^2}}\right) \tag{3.3.30}$$

which actually looks reasonable.

The previous problem illustrates addition of angular momentum. If we have two particles with spin j_1 and j_2, then we can express the eigenstates of total angular momentum $|J, M\rangle$ in terms of the tensor products $|j_1, m_1\rangle \otimes |j_2, m_2\rangle$ of the single-particle angular momentum states. This representation is typically written in terms of Clebsch-Gordon coefficients $\langle j_1, m_1, j_2, m_2|J, M\rangle$ as

$$|J, M\rangle = \sum_{m_1+m_2=M} \langle j_1, m_1, j_2, m_2|J, M\rangle \, |j_1, m_1\rangle \otimes |j_2, m_2\rangle \, . \tag{3.3.31}$$

These Clebsch-Gordon coefficients, or "clebsches," can be computed using the highest weight method. First we start from $|j_1, j_1\rangle \otimes |j_2, j_2\rangle$ and use the lowering operator to construct $2(j_1+j_2)+1$ states (keeping track of normalization to get the clebsches). Included in this multiplet is one state with $J_z = j_1 + j_2 - 1$, but there are two total such states, so we take the other one and build a new multiplet, this one with $2(j_1 + j_2 - 1) + 1$ states. We continue this process all the way until we start from the lowest-highest-weight, $J_z = |j_1 - j_2|$, and build its multiplet.

In summary, if we add two particles with spins j_1 and j_2, the total system can have spin $j_1 + j_2$, $j_1 + j_2 - 1, \ldots, |j_1 - j_2|$. As a statement in representation theory, this is written

$$j_1 \otimes j_2 = (j_1 + j_2) \oplus (j_1 + j_2 - 1) \oplus \cdots \oplus |j_1 - j_2|, \tag{3.3.32}$$

where the representation of dimension $2j + 1$ is labeled by j.

Problem 3.13 (M13Q3)
Four spin-S spins are located at the corners of a square and interact antiferromagnetically ($J > 0$). Use the convention where the spin operators are dimensionless. The Hamiltonian is

$$H = J\left(\boldsymbol{S}_1 \cdot \boldsymbol{S}_2 + \boldsymbol{S}_2 \cdot \boldsymbol{S}_3 + \boldsymbol{S}_3 \cdot \boldsymbol{S}_4 + \boldsymbol{S}_4 \cdot \boldsymbol{S}_1\right). \tag{3.3.33}$$

a) What are a complete set of good quantum numbers that can be used to fully classify all of the eigenstates of H?

b) For spin-1/2 give the eigenenergy and the degeneracy of each energy level.

c) For general spin S, what are the energy, degeneracy, and quantum numbers of the ground state?

We start by rewriting two of the dot products in terms of squared spin operators:

$$H = \frac{J}{2}\left((\boldsymbol{S}_1 + \boldsymbol{S}_2)^2 + (\boldsymbol{S}_3 + \boldsymbol{S}_4)^2 + 2\boldsymbol{S}_2 \cdot \boldsymbol{S}_3 + 2\boldsymbol{S}_4 \cdot \boldsymbol{S}_1 - 4\hbar^2 S(S+1)\right), \tag{3.3.34}$$

where the last term comes from $-\boldsymbol{S}_1^2 - \cdots - \boldsymbol{S}_4^2$. We can then rewrite this as

$$H = \frac{J}{2}\left((\boldsymbol{S}_1 + \boldsymbol{S}_2 + \boldsymbol{S}_3 + \boldsymbol{S}_4)^2 - 2\boldsymbol{S}_1 \cdot \boldsymbol{S}_3 - 2\boldsymbol{S}_2 \cdot \boldsymbol{S}_4 - 4\hbar^2 S(S+1)\right). \tag{3.3.35}$$

Finally, we rewrite the remaining dot products and find

$$H = \frac{J}{2}\left((\boldsymbol{S}_1 + \boldsymbol{S}_2 + \boldsymbol{S}_3 + \boldsymbol{S}_4)^2 - (\boldsymbol{S}_1 + \boldsymbol{S}_3)^2 - (\boldsymbol{S}_2 + \boldsymbol{S}_4)^2\right). \tag{3.3.36}$$

Thus, the quantum numbers we need are $\boldsymbol{S}_{\text{tot}}^2$, $\boldsymbol{S}_{13}^2$, and $\boldsymbol{S}_{24}^2$, where these variables have the obvious meanings. The values of S_{tot} are given by the sum rule applied to the values of S_{13} and S_{24}.

If $S = 1/2$, then we have the following states:

S_{13}	S_{24}	S_{tot}	H	Degeneracy
0	0	0	0	1
1	0	1	0	3
0	1	1	0	3
1	1	0	$-2J\hbar^2$	1
1	1	1	$-J\hbar^2$	3
1	1	2	$J\hbar^2$	5

Looking only at the energy column, we see there is one state with $H = -2J\hbar^2$, three states with $H = -J\hbar^2$, seven states with $H = 0$, and five states with $H = J\hbar^2$.

The ground state will always be the one with $S_{13} = S_{24} = 2S$ and $S_{\text{tot}} = 0$. This gives

$$E_0 = -2J\hbar^2 S(2S+1). \tag{3.3.37}$$

It is a singlet state since $S_{\text{tot}} = 0$.

Apparently some people actually do experiments with this stuff, and in experiments it's a bit easier to measure an angle than it is to measure an angular momentum eigenstate. So, let's look at all this from a more concrete bent. A rotationally invariant Hamiltonian looks like

$$H = \frac{\boldsymbol{p}^2}{2m} + V(r) = -\frac{\hbar^2}{2m}\nabla^2 + V(r). \tag{3.3.38}$$

Writing ∇^2 in spherical coordinates, the Schrödinger equation becomes

$$\left(\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial^2\phi} + \frac{2m(E - V(r))}{\hbar^2}\right)\psi(r,\theta,\phi) = 0. \tag{3.3.39}$$

That's a little gross, but separation of variables helps a lot. Let $\psi(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi)$. First

we find that $\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = \text{const}$, and single-valuedness requires $\Phi(\phi) = e^{im\phi}$ with $m \in \mathbb{Z}$. Next we find

$$\sin\theta \frac{d}{d\theta}\left(\sin\theta \frac{d}{d\theta}\Theta\right) = \left(\text{const} \times \sin^2\theta - m^2\right)\Theta(\theta). \tag{3.3.40}$$

The solutions to this differential equation are the associated Legendre polynomials evaluated at $\cos\theta$: $\Theta(\theta) = P_\ell^m(\cos\theta)$. The solutions are well-defined and nonzero only for ℓ an integer, with the constant in the above equation becoming $\ell(\ell+1)$, and $-\ell \le m \le \ell$. Unsurprisingly, these are the same conditions that apply to the states $|\ell, m\rangle$. The complete angular part of the solution is called a spherical harmonic:

$$Y_{\ell,m}(\theta,\phi) = N_{\ell,m} e^{im\phi} P_\ell^m(\cos\theta), \tag{3.3.41}$$

where the normalization $N_{\ell,m}$ is chosen to make these functions orthonormal on the unit sphere.

The final equation is the radial equation,

$$\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) + \left(\frac{2m(E - V(r))r^2}{\hbar^2} - \ell(\ell+1)\right) R = 0. \tag{3.3.42}$$

If we let $R(r) = \frac{u(r)}{r}$, this becomes

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + \left(V(r) + \frac{\hbar^2\ell(\ell+1)}{2mr^2}\right)u = Eu. \tag{3.3.43}$$

This is the Schrödinger equation for a particle in one dimension with effective potential

$$V_{\text{eff}}(r) = V(r) + \frac{\hbar^2\ell(\ell+1)}{2mr^2}, \tag{3.3.44}$$

in exact analogy to the $\frac{L^2}{2mr^2}$ added to the potential in classical mechanics. Solving this equation completes the solution of a spherical quantum system.

Problem 3.14 (J02Q1)
A particle of mass m moves in the spherically symmetrical potential in 3 dimensions:

$$V(r) = \begin{cases} 0, & 0 \le r < a \\ -U_0, & a < r < b \\ 0, & b < r \end{cases} \tag{3.3.45}$$

where $U_0 > 0$.

What is the condition on U_0 so that there will not be any bound states?

We can immediately write down the radial wave equation,

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + \left(V(r) + \frac{\hbar^2\ell(\ell+1)}{2mr^2}\right)u(r) = Eu(r). \tag{3.3.46}$$

Since we are looking for the lowest-energy states, we can take $\ell = 0$, so we have

$$\frac{d^2u}{dr^2} = -\frac{2m(E+U_0)}{\hbar^2}u \quad (a \le r \le b), \qquad \frac{d^2u}{dr^2} = -\frac{2mE}{\hbar^2}u \quad (r < a \quad \text{or} \quad r > b). \tag{3.3.47}$$

We want $E < 0$, so let $\kappa = \sqrt{-\frac{2mE}{\hbar^2}}$. Then we have $u = Ae^{-\kappa r}$ for $r > b$ and $u = B\sinh(\kappa r)$ for $r < a$ (since $u \to 0$ as $r \to 0, \infty$). In the intermediate region we have

$$u(a \le r \le b) = C\cos(\lambda(r-a)) + D\sin(\lambda(r-a)), \tag{3.3.48}$$

where $\lambda = \sqrt{\frac{2m(U_0-E)}{\hbar^2}}$. The boundary conditions require

$$A\sinh(\kappa a) - C = 0, \tag{3.3.49}$$
$$A\kappa\cosh(\kappa a) - \lambda D = 0, \tag{3.3.50}$$
$$-Be^{\kappa B} + C\cos(\lambda(b-a)) + D\sin(\lambda(b-a)) = 0, \tag{3.3.51}$$
$$\kappa Be^{-\kappa b} - \lambda C\sin(\lambda(b-a)) + \lambda D\cos(\lambda(b-a)) = 0. \tag{3.3.52}$$

In order for this system to have a nontrivial solution, we must have

$$\det\begin{pmatrix} \sinh(\kappa a) & 0 & -1 & 0 \\ \kappa\cosh(\kappa a) & 0 & 0 & -\lambda \\ 0 & -e^{\kappa b} & \cos(\lambda(b-a)) & \sin(\lambda(b-a)) \\ 0 & \kappa e^{-\kappa b} & -\lambda\sin(\lambda(b-a)) & \lambda\cos(\lambda(b-a)) \end{pmatrix} = 0. \tag{3.3.53}$$

For any fixed U_0, the left hand side is a function of E. Presumably for some small enough U_0, this function has no roots less than zero, and as we increase U_0, eventually it develops a negative root. Let's assume the roots are continuous in U_0. Near the threshold U_0, the negative root should be very small, meaning κ is very small and $\lambda \approx \sqrt{\frac{2mU_0}{\hbar^2}}$. The equation reduces to

$$0 = \det\begin{pmatrix} \kappa a & 0 & -1 & 0 \\ \kappa & 0 & 0 & -\lambda \\ 0 & -1 & \cos(\lambda(b-a)) & \sin(\lambda(b-a)) \\ 0 & 0 & -\lambda\sin(\lambda(b-a)) & \lambda\cos(\lambda(b-a)) \end{pmatrix} \tag{3.3.54}$$

$$= \kappa a\det\begin{pmatrix} 0 & 0 & -\lambda \\ -1 & \cos(\lambda(b-a)) & \sin(\lambda(b-a)) \\ 0 & -\lambda\sin(\lambda(b-a)) & \lambda\cos(\lambda(b-a)) \end{pmatrix} - \kappa\det\begin{pmatrix} 0 & -1 & 0 \\ -1 & \cos(\lambda(b-a)) & \sin(\lambda(b-a)) \\ 0 & -\lambda\sin(\lambda(b-a)) & \lambda\cos(\lambda(b-a)) \end{pmatrix} \tag{3.3.55}$$

$$= \kappa a\det\begin{pmatrix} 0 & -\lambda \\ -\lambda\sin(\lambda(b-a)) & \lambda\cos(\lambda(b-a)) \end{pmatrix} - \kappa\det\begin{pmatrix} -1 & 0 \\ -\lambda\sin(\lambda(b-a)) & \lambda\cos(\lambda(b-a)) \end{pmatrix} \tag{3.3.56}$$

$$= -\kappa\lambda^2 a\sin(\lambda(b-a)) + \kappa\lambda\cos(\lambda(b-a)). \tag{3.3.57}$$

where we have expanded by minors in each step. We find that the threshold U_0 corresponds to the first solution of $\lambda a = \cot(\lambda(b-a))$. Letting λ_0 denote this first solution, the condition for no bound states is

$$U_0 < \frac{\hbar^2\lambda_0^2}{2m}. \tag{3.3.58}$$

Going back to the spherical harmonics, we haven't yet seen what they look like. We were able to construct the $|\ell, m\rangle$ states using the raising and lowering operators, so we can translate this

method into the coordinate basis to construct the $Y_{\ell,m}(\theta,\phi)$. Using the orbital angular momentum $\boldsymbol{L} = -i\hbar \boldsymbol{r} \times \nabla$, we find

$$L_z = -i\hbar \frac{\partial}{\partial \phi}, \qquad L_\pm = \hbar e^{\pm i\phi} \left(\pm \frac{\partial}{\partial \theta} + i \cot\theta \frac{\partial}{\partial \phi} \right). \tag{3.3.59}$$

The highest-weight state is represented by $Y_{\ell,\ell}(\theta,\phi)$. It must be an eigenfunction of L_z with eigenvalue $\ell\hbar$, so it has an overall factor of $e^{i\ell\phi}$. Additionally, it must be annihilated by L_+, which fixes $Y_{\ell,\ell}(\theta,\phi) \propto e^{i\ell\phi} \sin^\ell \phi$. The other states can be generated with L_-.

Problem 3.15 (J03Q3)
An unpolarized nucleus of spin $S = 2$ decays into a nucleus of spin 0, plus two alpha particles, both having spin 0 and orbital angular momentum $L = 1$. We would like to predict the probability distribution of the angle between the directions of motion of the outgoing alphas. (Assuming the original nucleus is unpolarized, there are no other meaningful angles in the problem.)

a) As a first step, use the techniques of angular momentum addition to construct states of total angular momentum 2 out of two particles of orbital angular momentum 1. Put otherwise, find the normalized linear combinations of $Y_{1,m_1}(\theta_1,\phi_1) Y_{1,m_2}(\theta_2,\phi_2)$ that provide a basis of the total angular momentum 2 representation.

b) Next, compute the probability density $p(\theta_1,\phi_1;\theta_2,\phi_2)$ for the joint angular distribution of both alphas when both $\theta_1 = \theta_2 = \pi/2$, so both alphas lie in the plane perpendicular to the S_z-quantization axis. Recall that the original $S = 2$ nucleus is unpolarized (i.e. has equal probability of being in the 5 different S_z substates).

c) The density obtained in the preceding part only depends on the angle $\omega = \phi_1 - \phi_2$ between the two particles. Use the fact that even for general $\theta_{1,2}$ and $\phi_{1,2}$, the density p will only depend on the angle ω between the directions of the two alphas to compute the probability distribution of ω.

We will label states by $|m_1, m_2\rangle$. We know that the highest-weight state is $|1,1\rangle$, and acting repeatedly with the lowering operator gives the following five states:

$$S_z = 2: \qquad |1,1\rangle \tag{3.3.60}$$

$$S_z = 1: \qquad \frac{1}{\sqrt{2}} \left(|1,0\rangle + |0,1\rangle \right) \tag{3.3.61}$$

$$S_z = 0: \qquad \frac{1}{\sqrt{6}} \left(|1,-1\rangle + 2\,|0,0\rangle + |-1,1\rangle \right) \tag{3.3.62}$$

$$S_z = -1: \qquad \frac{1}{\sqrt{2}} \left(|0,-1\rangle + |-1,0\rangle \right) \tag{3.3.63}$$

$$S_z = -2: \qquad |-1,-1\rangle\,. \tag{3.3.64}$$

For part (b), we need the spherical harmonics at $\ell = 1$. Using the method outlined above, we can compute

$$Y_{1,1}(\theta,\phi) = \sqrt{\frac{3}{8\pi}} e^{i\phi} \sin\theta, \tag{3.3.65}$$

$$Y_{1,0}(\theta,\phi) = -\sqrt{\frac{3}{4\pi}} \cos\theta, \tag{3.3.66}$$

$$Y_{1,-1}(\theta,\phi) = -\sqrt{\frac{3}{8\pi}} e^{-i\phi} \sin\theta. \tag{3.3.67}$$

At $\theta = \pi/2$, these functions are $\sqrt{\frac{3}{8\pi}} e^{i\phi}, 0, -\sqrt{\frac{3}{8\pi}} e^{-i\phi}$. We are given that the $S = 2$ nucleus has equal probability of being in any of the five S_z states above, and we take this probability to be statistical (rather than a quantum superposition, in which case we would need more information to define the state), so we have

$$p\left(\frac{\pi}{2}, \phi_1; \frac{\pi}{2}, \phi_2\right) = \frac{3}{40\pi}\left(\left|e^{i(\phi_1+\phi_2)}\right|^2 + \left|\frac{1}{\sqrt{6}}\left(e^{i(\phi_1-\phi_2)} + e^{i(\phi_2-\phi_1)}\right)\right|^2 + \left|e^{-i(\phi_1+\phi_2)}\right|^2\right) \tag{3.3.68}$$

$$= \frac{1}{20\pi}\left(3 + \cos^2(\phi_1 - \phi_2)\right). \tag{3.3.69}$$

Normalizing this gives

$$p(\omega) = \frac{2}{7\pi}\left(3 + \cos^2\omega\right). \tag{3.3.70}$$

3.4 Perturbation Theory

So we can solve harmonic oscillators and some simple spherically symmetric systems. This is good, because everything is a harmonic oscillator, but also not so good, because the previous claim is a lie. In general, solving a Hamiltonian is hard – this is why a lot of physicists have jobs. However, if a Hamiltonian is "close" in some sense to a Hamiltonian with known energies and eigenstates, then we can develop the energies and states of the full Hamiltonian in a perturbative expansion.

To see how this works, let H_0 be a Hamiltonian with known spectrum $|n^{(0)}\rangle$, with energies $H_0 |n^{(0)}\rangle = E_n^{(0)} |n^{(0)}\rangle$. Let the full Hamiltonian be

$$H = H_0 + \lambda V, \tag{3.4.1}$$

where V is some perturbing potential and λ is a parameter we can take arbitrarily small so that an expansion makes sense. We can express the (as of yet unknown) spectrum of H as

$$|n\rangle = |n^{(0)}\rangle + \lambda |n^{(1)}\rangle + \lambda^2 |n^{(2)}\rangle + \ldots, \tag{3.4.2}$$

$$E_n = E_n^{(0)} + \lambda E_n^{(1)} + \lambda^2 E_n^{(2)} + \ldots. \tag{3.4.3}$$

Now we write the Schrödinger equation. We have

$$H\,|n\rangle = E_n^{(0)}\,|n^{(0)}\rangle + \lambda\left(V\,|n^{(0)}\rangle + H_0\,|n^{(1)}\rangle\right) + \dots \tag{3.4.4}$$

$$E_n\,|n\rangle = E_n^{(0)}\,|n^{(0)}\rangle + \lambda\left(E_n^{(1)}\,|n^{(0)}\rangle + E_n^{(0)}\,|n^{(1)}\rangle\right) + \dots . \tag{3.4.5}$$

The last equation implies $\langle n^{(0)}|n^{(1)}\rangle$ is pure imaginary. Equating the λ terms in the first two equations and acting with $\langle n^{(0)}|$, we find

$$E_n^{(1)} = \langle n^{(0)}|V|n^{(0)}\rangle\,. \tag{3.4.6}$$

To find the first-order correction to the state, we use

$$V\,|n^{(0)}\rangle = \sum_k |k^{(0)}\rangle\,\langle k^{(0)}|V|n^{(0)}\rangle = \sum_{k\neq n} |k^{(0)}\rangle\,\langle k^{(0)}|V|n^{(0)}\rangle + E^{(1)}\,|n^{(0)}\rangle\,. \tag{3.4.7}$$

We then have, at order λ,

$$H_0\,|n^{(1)}\rangle + \sum_{k\neq n} |k^{(0)}\rangle\,\langle k^{(0)}|V|n^{(0)}\rangle = E_n^{(0)}\,|n^{(1)}\rangle\,. \tag{3.4.8}$$

If we expand $|n^{(1)}\rangle = \sum \alpha_k\,|k^{(0)}\rangle$, and assume that $|n^{(0)}\rangle$ is the only state with energy $E_n^{(0)}$ (more on the degenerate case later), this equation fixes the α_k coefficients and we have

$$|n^{(1)}\rangle = \sum_{k\neq n} \frac{\langle k^{(0)}|V|n^{(0)}\rangle}{E_n^{(0)} - E_k^{(0)}}\,|k^{(0)}\rangle\,. \tag{3.4.9}$$

Problem 3.16 (M10Q2)
A non-relativistic particle with mass m moves one-dimensionally in the potential

$$V(x) = \frac{1}{2}m\omega^2 x^2 + \lambda x^4, \qquad \text{with } \lambda > 0. \tag{3.4.10}$$

Let $|\Psi_0(\lambda)\rangle$ be the ground state of the system, and $E_0(\lambda)$ be the ground state energy. For small λ, the quartic term in the potential can be treated as a small perturbation of the $\lambda = 0$ harmonic oscillator problem, which has an oscillation frequency ω.

a) The particle coordinate x can be expressed as an operator in terms of $a^\dagger$ and a, the raising and lowering operators for the $\lambda = 0$ harmonic oscillator problem, where $a\,|\Psi_0(\lambda = 0)\rangle = 0$. Give such an expression for x.

b) Compute the perturbation expansion of the ground-state energy $E_0(\lambda)$ up to first order in λ.

c) Again up to first order in λ, compute the perturbation expansion of the ground-state expectation value $\langle\Psi_0(\lambda)|x^2|\Psi_0(\lambda)\rangle$.

   In the opposite limit of large positive $\lambda \to \infty$, the leading behavior of the ground state energy $E_0(\lambda)$ will be proportional to λ^α where α is a positive exponent.

d) (Up to an undetermined numerical multiplicative factor) find the asymptotic large-λ behavior of the ground-state energy $E_0(\lambda)$, giving the explicit value of α.

We have the usual expression for x,

$$x = \sqrt{\frac{\hbar}{2m\omega}}\left(a + a^\dagger\right). \tag{3.4.11}$$

The first order correction to the ground state energy is

$$E_0^{(1)} = \left(\frac{\hbar}{2m\omega}\right)^2 \left\langle 0 \left| \left(a + a^\dagger\right)^4 \right| 0 \right\rangle = 3\left(\frac{\hbar}{2m\omega}\right)^2, \tag{3.4.12}$$

and so we have

$$E_0(\lambda) = \frac{\hbar\omega}{2} + 3\lambda\left(\frac{\hbar}{2m\omega}\right)^2 + \mathcal{O}\left(\lambda^2\right). \tag{3.4.13}$$

The first-order correction to the ground state is

$$|\Psi_0^{(1)}\rangle = \left(\frac{\hbar}{2m\omega}\right)^2 \sum_{k\neq 0} \frac{\left\langle k \left| \left(a + a^\dagger\right)^4 \right| 0 \right\rangle}{-k\hbar\omega} |k\rangle \tag{3.4.14}$$

$$= \left(\frac{\hbar}{2m\omega}\right)^2 \left(-\frac{3\sqrt{2}}{\hbar\omega}|2\rangle - \frac{\sqrt{6}}{2\hbar\omega}|4\rangle\right). \tag{3.4.15}$$

It follows that

$$\langle\Psi_0(\lambda)|x^2|\Psi_0(\lambda)\rangle = \frac{\hbar}{2m\omega} - 3\lambda\frac{\hbar^2}{2m^3\omega^4}. \tag{3.4.16}$$

For λ very large, we can ignore the harmonic oscillator potential, so we have

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \lambda x^4. \tag{3.4.17}$$

The only way to form an energy out of these parameters is $\left(\frac{\hbar^4\lambda}{m^3}\right)^{1/3}$, so $\alpha = 1/3$.

Problem 3.17 (J98Q1)
A spin $\frac{1}{2}$ particle of mass m moves in a spherical harmonic oscillator potential and is also subject to a parity violating perturbation. The Hamiltonian is $H = H_0 + H_1$ with

$$H_0 = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 r^2, \qquad \text{and} \qquad H_1 = \lambda\boldsymbol{\sigma}\cdot\boldsymbol{r} \tag{3.4.18}$$

where σ_x, σ_y and σ_z are the Pauli spin matrices.

As a measure of the parity violation, compute the expectation value $\langle z\sigma_z\rangle$ for the ground state, to first order in λ.

We need to compute $\langle 0|z\sigma_z|0\rangle$. We might be concerned, since the ground state $|0\rangle$ actually has two-fold spin degeneracy. However, this degeneracy is not broken at first order in λ, so we actually don't have to worry about this.

Anyway, the first order contribution is

$$2\lambda \,\mathrm{Re}\left(\langle 0^{(0)}|z\sigma_z|0^{(1)}\rangle\right). \tag{3.4.19}$$

Substituting the first order correction to the state, we have

$$2\lambda \sum_{n\neq 0} \frac{|\langle 0|z\sigma_z|n\rangle|^2}{E_0 - E_n} = -\frac{2\lambda\sigma_z^2 x_0^2}{\hbar\omega} x_0^2 = -\frac{\lambda}{m\omega^2} I, \tag{3.4.20}$$

where I denotes the 2×2 identity matrix.

The second order correction to the energy can be found using the same approach. At order λ^2, the Schrödinger equation gives

$$E_n^{(2)}|n^{(2)}\rangle + E_n^{(1)}|n^{(1)}\rangle + E_n^{(2)}|n^{(0)}\rangle = V|n^{(1)}\rangle + H_0|n^{(2)}\rangle. \tag{3.4.21}$$

Acting on the left with $\langle n^{(0)}|$, this becomes

$$E_n^{(2)} = \sum_{k\neq n} \frac{|\langle k^{(0)}|V|n^{(0)}\rangle|^2}{E_n^{(0)} - E_k^{(0))}}. \tag{3.4.22}$$

I would be remiss if I didn't mention a diagrammatic representation for the perturbation to the energy. To get the correction at order λ^k, draw a circle and draw $k-1$ arrows in the circle. There are other rules, but at $k = 1, 2$ they don't matter. To evaluate the diagrams, think of each arc as a matrix element where sources of arrows are $|n\rangle$ and targets are $|m\rangle$ with $m \neq n$, and each arrow as an inverse energy difference:

$$E_n^{(1)} = \bigcirc = \langle n|V|n\rangle, \qquad E_n^{(2)} = \bigcirc\!\!\uparrow = \sum_{m\neq n} \frac{\langle n|V|m\rangle\langle m|V|n\rangle}{E_n - E_m}. \tag{3.4.23}$$

There seems to be a representation like this at all orders, but that's outside the scope of this...whatever this document is.

Problem 3.18 (J10Q1)
Consider an isotropic three-dimensional harmonic oscillator described by the rotationally-invariant Hamiltonian

$$H = \frac{\boldsymbol{p}^2}{2m} + \frac{m\omega^2}{2}\boldsymbol{x}^2. \tag{3.4.24}$$

a) i. What are the energies and degeneracies of the lowest three energy levels?

ii. Account for the degeneracies by classifying states in these levels into total angular momentum multiplets.

b) By how much does the ground state energy change under the influence of a perturbation of the form

$$H' = \lambda(\boldsymbol{b}\cdot\boldsymbol{x})^3 \tag{3.4.25}$$

where $\boldsymbol{b}$ is some fixed vector, and λ is small? Calculate the correction up to second order in λ.

Now suppose that the oscillating particle has charge q. At time $t = 0$, a weak uniform electric field $\boldsymbol{E}$ is switched on, which then slowly decays as $\boldsymbol{E}(t) = \boldsymbol{E}_0 e^{-t/\tau}$, with $\tau > 0$.

c) What is the probability (to leading order in $|\boldsymbol{E}_0|$) that a system originally in the ground state will be in an excited state at a much larger time $t \gg \tau$?

We can write the harmonic oscillator Hamiltonian as a sum of three number operators, and so the energy levels are

$$E_{n_1,n_2,n_3} = \hbar\omega\left(n_1 + n_2 + n_3 + \frac{3}{2}\right). \tag{3.4.26}$$

Clearly there is one ground state with $E_0 = \frac{3}{2}\hbar\omega$, three degenerate excited states with $E_1 = \frac{5}{2}\hbar\omega$, and six degenerate doubly excited states with $E_2 = \frac{7}{2}\hbar\omega$.

In terms of angular momentum multiplets, the first level is the singlet, the second level is the spin-1 triplet, and the third level comes from summing two spin-1 excitations: the two spins can be parallel (giving spin 2, with 5 levels) or antiparallel (giving an additional singlet).

The cubic perturbation doesn't couple the ground state to itself (as can be shown explicitly, or by parity considerations), so the leading correction to the ground state energy is second order. The ground state will be coupled to the states with total excitation 1 or 3. The matrix elements for representative states are

$$\langle 1,1,1|H'|0,0,0\rangle = (\lambda x_0^3)6b_x b_y b_z, \tag{3.4.27}$$

$$\langle 2,1,0|H'|0,0,0\rangle = (\lambda x_0^3)3\sqrt{2}b_x^2 b_y, \tag{3.4.28}$$

$$\langle 3,0,0|H'|0,0,0\rangle = (\lambda x_0^3)\sqrt{6}b_x^3, \tag{3.4.29}$$

$$\langle 1,0,0|H'|0,0,0\rangle = (\lambda x_0^3)(3b_x^3 + 3b_x b_y^2 + 3b_x b_z^2), \tag{3.4.30}$$

where $x_0 = \sqrt{\frac{\hbar}{2m\omega}}$, as can be shown by simple computations with the raising and lowering operators. We thus have

$$E_0^{(2)} = -\lambda^2 x_0^6\left(\frac{36b_x^2b_y^2b_z^2 + (18b_x^4b_y^2 + \ldots) + (6b_x^6 + \ldots)}{3\hbar\omega} + \frac{(2b_x^3 + 3b_x b_y^2 + 3b_x b_z^2)^2 + \ldots}{\hbar\omega}\right). \tag{3.4.31}$$

The dots refer to terms coming from the other states, e.g. $|0,1,0\rangle$. Collecting terms, we find

$$E_0 = \frac{3}{2}\hbar\omega - \frac{11\lambda^2 x_0^6}{\hbar\omega}|\boldsymbol{b}|^6 + \mathcal{O}(\lambda^3), \tag{3.4.32}$$

a miracle which was guaranteed from the outset by spherical symmetry (i.e., we can get the 11 from comparing the b_x^6 term, and the rest follows).

For part (c), we need time-dependent perturbation theory, so, spoilers. The perturbing Hamiltonian is

$$V = -e^{-t/\tau}\boldsymbol{E}_0 \cdot \boldsymbol{x}. \tag{3.4.33}$$

This only couples the ground state to $|1,0,0\rangle$ and similar, and the matrix element is

$$\langle 1,0,0|V|0,0,0\rangle = -e^{-t/\tau}E_{0,x}x_0. \tag{3.4.34}$$

Thus, we have

$$c^{(1)}_{1,0,0}(\infty) = \frac{i}{\hbar}E_{0,x}x_0\int_0^\infty e^{-i\omega t}e^{-t/\tau}\,dt = \frac{iE_{0,x}x_0}{\hbar(\tau^{-1}+i\omega)}, \tag{3.4.35}$$

so the leading-order probability of finishing in one of the three excited states is

$$P = |c^{(1)}_{1,0,0}(\infty)|^2 + \ldots = \frac{|\boldsymbol{E}_0|^2x_0^2\tau^2}{\hbar^2(1+\omega^2\tau^2)}. \tag{3.4.36}$$

Problem 3.19 (J05Q2)
A particle of charge e is free to move on a circular ring of radius R centered around a fixed particle of charge $+e$. The ring is in the xy plane. A uniform electric field E is applied in the x-direction.

a) Compute the ground state energy to leading order in small E.

b) Develop an approximation for the ground state energy in the limit of large E.

The wavefunction for the nth unperturbed state is $\psi_n(\theta) \sim e^{in\theta}$, which gives unperturbed energies $E_n = \frac{\hbar^2n^2}{2mR^2} - \frac{q^2}{R}$. The perturbing Hamiltonian can be written as $eER\cos\theta$, which has matrix elements

$$\langle n|eE\cos\theta|m\rangle = eER\int_0^{2\pi} e^{i(m-n)\theta}\cos\theta\,d\theta = \frac{eER}{2}(\delta_{m,n+1}+\delta_{m,n-1}). \tag{3.4.37}$$

Clearly there is no first-order shift to the ground state energy, and so we use the second order correction to find

$$E_0 = \frac{\hbar^2}{2mR^2} - \frac{q^2}{R} - \frac{e^2E^2R^4m}{6\hbar^2} + \mathcal{O}(E^3). \tag{3.4.38}$$

For large E, the ground state will be localized at a point on the circle, and so we can write the potential as $\frac{1}{2}eER\theta^2$. This looks like a harmonic oscillator, with ground state energy $\frac{1}{2}\hbar\omega = \frac{\hbar}{2}\sqrt{\frac{eER}{m}}$. Thus, in total, the ground state energy at large E will be $\frac{\hbar}{2}\sqrt{\frac{eER}{m}} - \frac{q^2}{R}$.

So much for nondegenerate perturbation theory. What if the Hamiltonian H has degenerate states

and the perturbation breaks the degeneracy? For example, consider the two-state system

$$H = E_0 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \lambda \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{3.4.39}$$

At $\lambda = 0$, both states have energy E_0. For $\lambda \neq 0$, the energies are $E_0 \pm \lambda$. However, we cannot obtain this result through nondegenerate perturbation theory, because we don't have a unique choice of the two eigenstates for the unperturbed Hamiltonian. It is clear how to fix this issue: when there is a degenerate eigenspace in the unperturbed Hamiltonian, we should choose a basis of eigenstates of the perturbation restricted to that eigenspace.

Problem 3.20 (M00Q1)
Consider a spin-$\frac{1}{2}$ particle constrained to move on a 1D line with a harmonic oscillator potential and a magnetic field so that the Hamiltonian is:

$$H = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 x^2 + \omega S_z. \tag{3.4.40}$$

The first energy level is not degenerate but all the other levels are doubly-degenerate.

Now add a small magnetic field in the $\hat{x}$ direction with a magnitude proportional to x. The Hamiltonian is:

$$H = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 x^2 + \omega S_z + \alpha x S_x. \tag{3.4.41}$$

Calculate the energy difference in the levels to lowest order.

To clarify: for the initial Hamiltonian, the degeneracy is between states $|n,\uparrow\rangle$ and $|n+1,\downarrow\rangle$, except of course for the ground state $|0,\downarrow\rangle$ which is not degenerate. We now need to understand how this degeneracy is split by the nonuniform magnetic field perturbation. The relevant matrix element is

$$\langle n+1,\downarrow | x S_x | n,\uparrow\rangle = \frac{\hbar x_0}{2}\sqrt{n+1}, \tag{3.4.42}$$

so the perturbing Hamiltonian looks like $\frac{\hbar x_0}{2}\sqrt{n+1}\sigma_x$ on this space. Using the eigenvectors of σ_x and applying first-order perturbation theory, we find that the two levels are split by $\alpha\hbar x_0\sqrt{n+1} = \alpha\sqrt{\frac{\hbar^3(n+1)}{2m\omega}}$.

3.5 Time-Dependent Perturbations

Things change over time. That's kind of the whole point of physics, I guess. So what happens when a perturbation changes over time?

There are some limiting cases. For example, a perturbation can turn on suddenly, much faster than any relevant timescales in the system. In this case, the state of the system does not change as a result of the perturbation (it doesn't have time to change), and so the old state evolves under the new Hamiltonian. We've already done problems like this.

In the opposite limit, a perturbation could change much more slowly than any relevant timescales; in this case we refer to the perturbation as adiabatic. Intuitively, if the Hamiltonian is changing very slowly, then the eigenstates also change slowly, and the system will have ample time to follow this change and stay in "the same" state. For example, if we start with the ground state of a harmonic oscillator and then adiabatically increase the frequency, we expect to end up in the ground state of the higher frequency harmonic oscillator.

This intuition is made precise in the adiabatic theorem. Recall that the time dependent Schrödinger equation $i\hbar\partial_t\psi = H\psi$ can be solved by finding eigenstates $H|\psi_n\rangle = E_n|\psi_n\rangle$, and then expanding the solution in these eigenstates as

$$|\psi(t)\rangle = \sum c_n e^{-iE_n t/\hbar}|\psi_n\rangle\,, \tag{3.5.1}$$

where $c_n = \langle\psi_n|\psi(0)\rangle$. Inspired by this approach, imagine solving the time-independent Schrödinger equation at all times,

$$H(t)|\psi_n(t)\rangle = E_n(t)|\psi_n(t)\rangle\,. \tag{3.5.2}$$

It's implied here that we're doing this in a reasonable way, i.e., $E_n(t)$ is continuous for all n. Moreover, we assume that energy levels don't cross – essentially, the system is nondegenerate at all times. Now we can try to write our state as

$$|\psi(t)\rangle = \sum c_n(t) e^{i\theta_n(t)}|\psi_n(t)\rangle\,, \tag{3.5.3}$$

where $\theta_n(t) = -\frac{1}{\hbar}\int_0^t E_n(t)\,dt$ is the generalization of the phase factor in the time-independent case. Substituting this state into the Schrödinger equation, we find

$$i\hbar\sum\left(\dot{c}_n(t)|\psi_n(t)\rangle + c_n(t)|\dot{\psi}_n(t)\rangle\right)e^{i\theta_n(t)} = 0. \tag{3.5.4}$$

At a fixed time the energy eigenstates form a complete basis, and so acting with $\langle\psi_m(t)|$ we find

$$\dot{c}_m(t) + \sum_n c_n(t)\langle\psi_m(t)|\dot{\psi}_n(t)\rangle\, e^{i(\theta_n(t)-\theta_m(t))} = 0. \tag{3.5.5}$$

So, we have a differential equation for the coefficients, but it depends on the inner products $\langle\psi_m(t)|\dot{\psi}_n(t)\rangle$. We know that $H(t)|\psi_n(t)\rangle = E_n(t)|\psi_n(t)\rangle$, and differentiating this, we find

$$\dot{H}(t)|\psi_n(t)\rangle + H(t)|\dot{\psi}_n(t)\rangle = \dot{E}_n(t)|\psi_n(t)\rangle + E_n(t)|\dot{\psi}_n(t)\rangle\,, \tag{3.5.6}$$

and so the inner product for $m \neq n$ is

$$\langle\psi_m(t)|\dot{\psi}_n(t)\rangle = \frac{\langle\psi_m(t)|\dot{H}(t)|\psi_n(t)\rangle}{E_n(t) - E_m(t)}. \tag{3.5.7}$$

Thus, we have

$$\dot{c}_m(t) + c_m(t)\langle\psi_m(t)|\dot{\psi}_m(t)\rangle + \sum_{n\neq m} c_n(t)\frac{\langle\psi_m(t)|\dot{H}(t)|\psi_n(t)\rangle}{E_n(t) - E_m(t)}e^{i(\theta_n(t)-\theta_m(t))} = 0. \tag{3.5.8}$$

Here comes the fun part. We've assumed the levels don't cross (and what the hell, let's assume their separation is bounded below by some finite energy), and we're assuming $\dot{H}(t)$ is as small as

we want it to be. That means the nasty sum goes away, and we're left with a linear equation which can be readily integrated to give

$$c_n(t) = c_n(0) \exp\left(\int_0^t \langle \psi_n(t) | \dot{\psi}_n(t) \rangle \, dt \right). \tag{3.5.9}$$

There are two important points here. First, if $c_n(0) = 0$, then it stays zero: adiabatic perturbations don't excite new energy levels, consistent with our intuition. Second, normalization requires that $\langle \psi_n(t) | \dot{\psi}_n(t) \rangle$ is pure imaginary, so we have a phase factor multiplying $c_n(0)$. This is the famous Berry phase.

Problem 3.21 (M01Q1)
A spin 1/2 particle with magnetic moment μ is fixed to a point in space. Let $|\uparrow\rangle$ and $|\downarrow\rangle$ denote the states with $S_z = \frac{1}{2}$ and $S_z = -\frac{1}{2}$. We turn on a constant magnetic field with magnitude B_0 and the direction given by:

$$\boldsymbol{B} = B_0 \left(\hat{\boldsymbol{x}} \sin\theta \cos\phi + \hat{\boldsymbol{y}} \sin\theta \sin\phi + \hat{\boldsymbol{z}} \cos\theta \right). \tag{3.5.10}$$

Here θ and ϕ are constant angles.

a) Find the ground state. Denote it by $|\theta, \phi\rangle$.

   Now we make ϕ change slowly with time $\phi = \omega t$.

b) In the adiabatic limit that ω is very small, the wavefunction can be approximated as

$$|t\rangle \sim e^{i\varphi(t)} \, |\theta, \omega t\rangle \, . \tag{3.5.11}$$

   Here $|\theta, \omega t\rangle$ is the state you found above. Find $\varphi(t)$. ($\varphi\left(\frac{2\pi}{\omega}\right)$ is called the Berry phase.)

   Suppose that at time $t = 0$ the particle is in the ground state $|\theta, 0\rangle$. Now we turn on the magnetic field for a whole cycle until time $t = \frac{2\pi}{\omega}$. At the end of the cycle we keep the magnetic field at the constant final value $\boldsymbol{B} = B_0(\hat{\boldsymbol{x}} \sin\theta + \hat{\boldsymbol{z}} \cos\theta)$.

c) Find the probability, to leading order in ω, that at the end of the cycle the particle will be in the excited state.

The potential is proportional to

$$-\boldsymbol{B} \cdot \boldsymbol{\sigma} = -B_0 \begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}. \tag{3.5.12}$$

The eigenvalues of this matrix are clearly ± 1, and the ground state will correspond to the $+1$ eigenstate because of the minus sign outside. This eigenstate is $|\theta, \phi\rangle = \cos\frac{\theta}{2} |\uparrow\rangle + \sin\frac{\theta}{2} e^{i\phi} |\downarrow\rangle$.

The Berry phase here is given by

$$\varphi(t) = \int_0^t \langle \theta, \omega t | \left(i\omega \sin\frac{\theta}{2} e^{i\omega t} |\downarrow\rangle \right) dt = i(\omega t) \sin^2\frac{\theta}{2}. \tag{3.5.13}$$

This means that after a full rotation, the state acquires a phase $\varphi\left(\frac{2\pi}{\omega}\right) = 2\pi \sin^2\frac{\theta}{2}$, so if $\theta \neq 0, \pi$ we will not return to the original state.

The adiabatic theorem tells us that the probability of transitioning to the excited state will vanish at first order in ω, so we have to be more detailed. Let $|0\rangle$ denote the ground state and $|1\rangle$ the excited state. We go back to the exact equation we derived, which in this case looks like

$$\dot{c}_1(t) + c_1(t)\left\langle 1\left|\frac{d}{dt}\right|1\right\rangle + c_0(t)\frac{\langle 1|\dot{H}(t)|0\rangle}{E_0 - E_1}e^{i(E_0-E_1)t/\hbar} = 0. \tag{3.5.14}$$

The energies are $E_1 = \mu B$ and $E_0 = -\mu B$, and the inner products are

$$\left\langle 1\left|\frac{d}{dt}\right|1\right\rangle = \left(\sin\frac{\theta}{2}e^{i\omega t}\langle\uparrow| - \cos\frac{\theta}{2}\langle\downarrow|\right)\left(-i\omega\sin\frac{\theta}{2}e^{-i\omega t}|\uparrow\rangle\right) = -i\omega\sin^2\frac{\theta}{2}, \tag{3.5.15}$$

$$\langle 1|\dot{H}(t)|0\rangle = \omega\mu B_0\sin\theta\,\langle 1\,|\,(\sigma_y\cos\omega t - \sigma_x\sin\omega t)\,|\,0\rangle = -ie^{i\omega t}\omega\mu B_0\sin\theta\cos\theta, \tag{3.5.16}$$

so we have

$$\dot{c}_1(t) - i\omega\sin^2\left(\frac{\theta}{2}\right)c_1(t) + c_0(t)\frac{i}{2}\omega\sin\theta\cos\theta e^{i(\omega-2\mu B_0/\hbar)t} = 0. \tag{3.5.17}$$

To zeroth order in ω we have $c_0(t) = \exp\left(i\omega t\sin^2\frac{\theta}{2}\right)$ and $c_1(t) = 0$. This equation then gives

$$c_1(t) = \frac{i\omega}{4}\sin 2\theta\int_0^t \exp\left(i\left(\omega - 2\mu B_0/\hbar + \omega\sin^2\frac{\theta}{2}\right)t\right)dt \tag{3.5.18}$$

$$= \frac{i\omega\sin 2\theta}{4\left(\omega - 2\mu B_0 + \omega\sin^2\frac{\theta}{2}\right)}\left(\exp\left(i\left(\omega - 2\mu B_0/\hbar + \omega\sin^2\frac{\theta}{2}\right)t\right) - 1\right). \tag{3.5.19}$$

The probability of transitioning to the excited state is

$$P = |c_1(2\pi/\omega)|^2 = \frac{\sin^2\theta\cos^2\theta}{\left(1 + \sin^2\frac{\theta}{2} - 2\mu B_0/\hbar\omega\right)^2}\sin^2\left(\pi\sin^2\frac{\theta}{2} - \frac{2\pi\mu B_0}{\hbar\omega}\right) \tag{3.5.20}$$

That last part was a little unwieldly, so let's make it wieldier. Specifically, it would be nice to do away with all these factors of $e^{iEt/\hbar}$, because, boring. So let's absorb those factors into the definition of the states. To be precise, let $H(t) = H_0 + V(t)$, that is, the unperturbed Hamiltonian H_0 is fixed. Then we define new "interaction picture" states

$$|\psi_n(t)\rangle_I = e^{iE_nt/\hbar}|\psi_n(t)\rangle, \tag{3.5.21}$$

or more generally, $|\psi(t)\rangle_I = e^{iH_0t/\hbar}|\psi(t)\rangle$. All this means is that the only time dependence of $|\psi_n(t)\rangle$ comes from $V(t)$. Another way to think of it is as a mix between the Schrödinger and Heisenberg formulations, where operators evolve according to H_0 in the Heisenberg sense while states evolve according to $V(t)$ in the Schrödinger sense. It's not that big a deal, but people tend to get really hot and bothered over this for some reason.

Anyway, with that out of the way, the Schrödinger equation in the interaction picture reads

$$i\hbar|\psi(t)\rangle_I = V_I(t)|\psi(t)\rangle_I, \tag{3.5.22}$$

where $V_I(t) = e^{iH_0t/\hbar}V(t)e^{-iH_0t/\hbar}$. If we expand in eigenstates $|\psi_n\rangle$ of H_0,

$$|\psi(t)\rangle_I = \sum c_n(t)|\psi_n\rangle, \tag{3.5.23}$$

then we find

$$i\hbar\dot{c}_n(t) = \sum c_m(t) \langle\psi_n|V_I(t)|\psi_m\rangle = \sum c_m(t) e^{i(E_m-E_n)t/\hbar} \langle\psi_n|V(t)|\psi_m\rangle . \tag{3.5.24}$$

This is at least a little bit nicer looking.

Problem 3.22 (J12Q1)
Consider a two-level system with two orthogonal states $|g\rangle$ and $|e\rangle$. It has the time-dependent Hamiltonian:

$$H(t) = \hbar\omega |e\rangle\langle e| + V\cos(\omega t)\left(|e\rangle\langle g| + |g\rangle\langle e|\right). \tag{3.5.25}$$

Assume that the time-dependent term is small, $\hbar\omega \gg V > 0$, so that you may make the corresponding approximation. At time $t = 0$ the state of the system is specified by the initial complex amplitudes c_{g0} and c_{e0}:

$$|\psi(t=0)\rangle = c_{g0}|g\rangle + c_{e0}|e\rangle . \tag{3.5.26}$$

What is the state of the system $|\psi(t)\rangle$ at other times t?

For the coefficients in the interaction picture, we may immediately write

$$i\hbar\dot{c}_g(t) = c_e(t)e^{i\omega t}V\cos(\omega t) = Vc_e(t)\frac{e^{2i\omega t}+1}{2}, \tag{3.5.27}$$

$$i\hbar\dot{c}_e(t) = c_g(t)e^{-i\omega t}V\cos(\omega t) = Vc_g(t)\frac{e^{-2i\omega t}+1}{2}. \tag{3.5.28}$$

We can formally solve this system by writing

$$\begin{pmatrix} c_g \\ c_e \end{pmatrix} = \exp\left(-\frac{iV}{\hbar}\int_0^t \begin{pmatrix} 0 & \frac{e^{2i\omega t}+1}{2} \\ \frac{e^{-2i\omega t}+1}{2} & 0 \end{pmatrix} dt\right)\begin{pmatrix} c_{g0} \\ c_{e0} \end{pmatrix}. \tag{3.5.29}$$

Now we have to make this a little less formal. The bowties are coming off. We have

$$\int_0^t \begin{pmatrix} 0 & \frac{e^{2i\omega t}+1}{2} \\ \frac{e^{-2i\omega t}+1}{2} & 0 \end{pmatrix} dt = \begin{pmatrix} 0 & \frac{t}{2} + \frac{1}{2\omega}e^{i\omega t}\sin\omega t \\ \frac{t}{2} + \frac{1}{2\omega}e^{-i\omega t}\sin\omega t & 0 \end{pmatrix}. \tag{3.5.30}$$

Evidently this is Hermitian, a good check on the arithmetic so far. But the exact arithmetic going forward does not look pleasant (though it is completely doable), so it's time to use the approximation $\frac{V}{\hbar\omega} \ll 1$ to ignore the second terms, so we have

$$\begin{pmatrix} c_g \\ c_e \end{pmatrix} = \exp\left(-\frac{iVt}{2\hbar}\sigma_x\right)\begin{pmatrix} c_{g0} \\ c_{e0} \end{pmatrix} \tag{3.5.31}$$

$$= e^{-\frac{iVt}{2\hbar}}\frac{c_{g0}+c_{e0}}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{\frac{iVt}{2\hbar}}\frac{c_{g0}-c_{e0}}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{3.5.32}$$

$$= \begin{pmatrix} c_{g0}\cos\left(\frac{Vt}{2\hbar}\right) - ic_{e0}\sin\left(\frac{Vt}{2\hbar}\right) \\ c_{e0}\cos\left(\frac{Vt}{2\hbar}\right) - ic_{g0}\sin\left(\frac{Vt}{2\hbar}\right) \end{pmatrix} \tag{3.5.33}$$

This is in the interaction picture, so $c_e(t)$ is the coefficient of $e^{i\omega t}|e\rangle$. Thus, the state is

$$|\psi(t)\rangle = \left(c_{g0}\cos\left(\frac{Vt}{2\hbar}\right) - ic_{e0}\sin\left(\frac{Vt}{2\hbar}\right)\right)|g\rangle + e^{i\omega t}\left(c_{e0}\cos\left(\frac{Vt}{2\hbar}\right) - ic_{g0}\sin\left(\frac{Vt}{2\hbar}\right)\right)|e\rangle . \tag{3.5.34}$$

Let's come up with a more systematic way of taking advantage of the smallness of a perturbing potential. The Schrödinger equation in the form $i\hbar\partial_t |\psi(t)\rangle_I = V_I |\psi(t)\rangle_I$ implies that if we let $|\psi(t)\rangle_I = U(t) |\psi(0)\rangle_I$, then

$$i\hbar\partial_t U(t) = V_I(t)U(t). \tag{3.5.35}$$

Taking an integral we have

$$U(t) = -\frac{i}{\hbar}\int_0^t dt'\, V_I(t')U(t'). \tag{3.5.36}$$

To zeroth order in the potential, $U(t) = 1$. Using this equation, we can recursively obtain all higher orders:

$$U^{(1)}(t) = -\frac{i}{\hbar}\int_0^t V_I(t')\, dt', \tag{3.5.37}$$

$$U^{(2)}(t) = \left(-\frac{i}{\hbar}\right)^2 \int_0^t dt'\, V_I(t') \int_0^{t'} dt''\, V_I(t''), \tag{3.5.38}$$

$$\tag{3.5.39}$$

and so on. To be fancy, we can write this as

$$U(t) = T\exp\left(-\frac{i}{\hbar}\int_0^t V_I(t')\, dt'\right), \tag{3.5.40}$$

where T stands for time-ordering.

We can use this fancypants expansion for the time evolution operator to work out approximations for the coefficients $c_n(t)$. We have

$$c_n(t) = \langle\psi_n(t)|U(t)|\psi(0)\rangle\,. \tag{3.5.41}$$

Let's say the system starts in state m, and some perturbation $V(t)$ is applied for a finite time. Then the amplitude of it ending up in state n would be

$$c_n(t) = \langle\psi_n|U(t)|\psi_m\rangle = \delta_{nm} - \frac{i}{\hbar}\int \langle\psi_n|V_I(t)|\psi_m\rangle\, dt + \mathcal{O}(V^2) \tag{3.5.42}$$

$$= \delta_{nm} - \frac{i}{\hbar}\int e^{i(E_n-E_m)t/\hbar}\,\langle\psi_n|V(t)|\psi_m\rangle\, dt + \mathcal{O}(V^2). \tag{3.5.43}$$

And voilá (is that accent in the right place?), there's our approximation.

Problem 3.23 (J03Q2)
An isolated hydrogen atom in the 2s level has a very long lifetime for radioactive decay because selection rules pretty much force it to decay by two-photon emission. In realistic situations, the atom suffers collisions that push the 2s level into the 2p levels, from which it rapidly decays by standard electric dipole emission.

In plasmas, the collisions are with ions that briefly subject the hydrogen atom to an electric field. Let us study what happens when an ion of charge Q, moving at constant velocity v passes by the H atom, making a closest distance of approach b. The electron in the atom sees a time-dependent potential

$$V_1(\boldsymbol{x}, t) = \frac{Qe}{|\boldsymbol{b} + \boldsymbol{v}t - \boldsymbol{x}|}, \qquad \boldsymbol{b}\cdot\boldsymbol{v} = 0. \tag{3.5.44}$$

Because the ion passes far from the atom, you can treat x as small and expand in powers of x. Keep the term of first order in x and treat it as the perturbing potential that induces transitions between the degenerate states of the $n = 2$ levels of hydrogen (the zeroth-order term doesn't depend on x and causes no transitions).

Use first-order time-dependent perturbation theory to find the transition amplitude for an atom originally in the 2s level to wind up in one of the 2p levels.

You will need some hydrogen wave functions:

$$\phi_{2s} = \frac{1}{2\sqrt{2\pi a_B^3}}(1 - r/2a_B)e^{-r/2a_B}, \tag{3.5.45}$$

$$\phi_{2p,0} = \frac{z}{4\sqrt{2\pi a_B^5}}e^{-r/2a_B}, \tag{3.5.46}$$

$$\phi_{2p,\pm1} = \frac{x \pm iy}{8\sqrt{2\pi a_B^5}}e^{-r/2a_B}. \tag{3.5.47}$$

This is...a weird mix of pretty cool and pretty sadistic. But here we go. To first order in x we have

$$V_1(\boldsymbol{x}, t) = \frac{Qe}{|\boldsymbol{b} + \boldsymbol{v}t|}\left(1 + \frac{x\cos\psi}{|\boldsymbol{b} + \boldsymbol{v}t|}\right), \tag{3.5.48}$$

where $\psi = 0$ is the direction of $\boldsymbol{b} + \boldsymbol{v}t$. We first compute the matrix elements. We have

$$\langle\phi_{2p,0}|V_1|\phi_{2s}\rangle = \frac{1}{16\pi a_B^4}\frac{Qe}{|\boldsymbol{b} + \boldsymbol{v}t|^2}\int d^3\boldsymbol{r}\left((1 - r/2a_B)r^2\cos\theta\cos\psi e^{-r/a_B}\right). \tag{3.5.49}$$

The hitch in the wagon is $\cos\psi$. We can express this in spherical coordinates by noting that

$$\cos\psi = \frac{\boldsymbol{x}\cdot(\boldsymbol{b} - \boldsymbol{v}t)}{x|\boldsymbol{b} - \boldsymbol{v}t|} = \frac{(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)\cdot(\boldsymbol{b} - \boldsymbol{v}t)}{|\boldsymbol{b} - \boldsymbol{v}t|}. \tag{3.5.50}$$

The $\cos\phi$ and $\sin\phi$ terms will vanish under the integral, so we have

$$\begin{aligned}
\langle\phi_{2p,0}|V_1|\phi_{2s}\rangle &= \frac{1}{16\pi a_B^4}\frac{Qe(\boldsymbol{b} + \boldsymbol{v}t)\cdot\hat{\boldsymbol{z}}}{|\boldsymbol{b} + \boldsymbol{v}t|^3}\int d^3\boldsymbol{r}\left((1 - r/2a_B)r^2\cos^2\theta e^{-r/a_B}\right) \\
&= \frac{1}{8a_B^4}\frac{Qe(\boldsymbol{b} + \boldsymbol{v}t)\cdot\hat{\boldsymbol{z}}}{|\boldsymbol{b} + \boldsymbol{v}t|^3}\int_0^\infty r^2\,dr\int_{-1}^1 d(\cos\theta)\left((1 - r/2a_B)r^2\cos^2\theta e^{-r/a_B}\right) \\
&= \frac{1}{12a_B^4}\frac{Qe(\boldsymbol{b} + \boldsymbol{v}t)\cdot\hat{\boldsymbol{z}}}{|\boldsymbol{b} + \boldsymbol{v}t|^3}\int_0^\infty r^2\,dr\left((1 - r/2a_B)r^2e^{-r/a_B}\right) \\
&= -3a_B\frac{Qe(\boldsymbol{b} + \boldsymbol{v}t)\cdot\hat{\boldsymbol{z}}}{|\boldsymbol{b} + \boldsymbol{v}t|^3}.
\end{aligned} \tag{3.5.51}$$

The other matrix elements follow similarly. We have

$$\langle\phi_{2p,\pm1}|V_1|\phi_{2s}\rangle = -\frac{3}{2}a_B\frac{Qe(\boldsymbol{b} + \boldsymbol{v}t)\cdot(\hat{\boldsymbol{x}} \pm i\hat{\boldsymbol{y}})}{|\boldsymbol{b} + \boldsymbol{v}t|^3}. \tag{3.5.52}$$

Now we have to integrate these. The key ingredient is

$$\int_{-\infty}^{\infty} \frac{dt}{|\boldsymbol{b}+\boldsymbol{v}t|^3} = \int_{-\infty}^{\infty} \frac{dt}{(b^2+(vt)^2)^{3/2}} = \frac{2}{vb^2}, \tag{3.5.53}$$

from which we find

$$c_{2p,0} = \frac{6iQea_B(\boldsymbol{b}\cdot\hat{\boldsymbol{z}})}{\hbar v b^2}, \tag{3.5.54}$$

$$c_{2p,\pm1} = \frac{3iQea_B(\boldsymbol{b}\cdot(\hat{\boldsymbol{x}}\pm i\hat{\boldsymbol{y}}))}{\hbar v b^2}. \tag{3.5.55}$$

These are the transition amplitudes. The probability of the atom ending up in one of the $2p$ states is

$$P = \sum_{m=-1}^{1} |c_{2p,m}|^2 = \left(\frac{6Qea_B}{\hbar v b^2}\right)^2 \left((\boldsymbol{b}\cdot\hat{\boldsymbol{z}})^2 + (\boldsymbol{b}\cdot\hat{\boldsymbol{x}})^2 + (\boldsymbol{b}\cdot\hat{\boldsymbol{y}})^2\right) = \left(\frac{6Qea_B}{\hbar v b}\right)^2. \tag{3.5.56}$$

Problem 3.24 (J11Q2)
A particle of mass m is confined to the interval $[0, L]$ by a one-dimensional infinite square well. It is initially in the ground state of the Hamiltonian with the confining potential.

a) At time $t = 0$ the potential within the well is suddenly changed to:

$$V(x) = \begin{cases} V_0, & \text{for } 0 < x < L/2, \\ 0 & \text{for } L/2 < x < L \end{cases}, \tag{3.5.57}$$

with $V_0 \ll E_1 - E_0$ (the latter being the gap between the two lowest states of the initial operator). The perturbation lasts for time T, after which the potential is restored to its initial value. What is the probability that after the potential is restored the particle's energy is E_1, calculated to first order in $V_0/(E_1 - E_0)$?

b) In a second experiment the value of V_0 (in the perturbing potential, as above) is increased very slowly, and to a much higher value $V \gg E_1$. It is switched off instantaneously when that value is reached. What is the probability that at this point the particle will have the energy E_1.

The wavefunctions for the two lowest states are

$$\psi_0(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{\pi x}{L}\right), \qquad \psi_1(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{2\pi x}{L}\right). \tag{3.5.58}$$

The relevant matrix element is

$$\langle 1|V|0\rangle = \frac{2V_0}{L}\int_0^{L/2} \sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{2\pi x}{L}\right) dx = \frac{4V_0}{3\pi}. \tag{3.5.59}$$

This gives

$$c_1(t>T) = -\frac{4iV_0}{3\pi\hbar}\int_0^T e^{i(E_1-E_0)t/\hbar}\,dt = -ie^{i(E_1-E_0)t/2\hbar}\frac{8V_0}{3\pi(E_1-E_0)}\sin\left(\frac{(E_1-E_0)t}{2\hbar}\right). \tag{3.5.60}$$

The probability is then

$$P = |c_1(t>T)|^2 = \left(\frac{8V_0}{3\pi(E_1-E_0)}\right)^2\sin^2\left(\frac{(E_1-E_0)t}{2\hbar}\right). \tag{3.5.61}$$

In the second experiment we have an adiabatic perturbation, and so the system will end up in the ground state of the new potential with $V_0 \gg E_1$. This ground state will be approximately

$$\psi_0'(x) = \sqrt{\frac{4}{L}}\sin\left(\frac{\pi(x-L/2)}{L/2}\right) \tag{3.5.62}$$

and so the probability is

$$P = |\,\langle 1|0'\rangle\,|^2 = \left(\frac{2\sqrt{2}}{L}\int_{L/2}^{L}\sin\left(\frac{2\pi x}{L}\right)\sin\left(\frac{2\pi x}{L}-\pi\right)\,dx\right)^2 = \frac{1}{2}. \tag{3.5.63}$$

Problem 3.25 (M05Q1)a) A spatially uniform, time-independent electric field $\boldsymbol{E}$ is applied to a hydrogen atom for which the electron was initially in its ground state $\phi_g = e^{-r}/\sqrt{\pi}$. The field is much too small to ionize the atom in any reasonable time, but it slightly distorts the electron charge distribution. The distorted electron wave function is $\psi = \psi(\boldsymbol{r})$, where $\boldsymbol{r}$ is the displacement (in units of the Bohr radius a_B) of the electron from the nucleus. You can ignore the spins of the electron and the nucleus. Since the polarized atom experiences no net force in the uniform field $\boldsymbol{E}$, the quantum-mechanical expectation value $\boldsymbol{E}' = (e/a_B^2)\,\langle\psi|\boldsymbol{r}/r^3|\psi\rangle$, of the electric field produced at the nucleus by the distorted electron charge distribution must cancel the applied field. Prove that this is so.

b) The applied electric field oscillates along the z axis with amplitude $E\cos\omega_{eg}t$ where $\omega_{eg} = (E_e - E_g)/\hbar$ is the Bohr frequency of the transition from the 1s ground state, of energy E_g, to the 2p excited state, of energy E_e. Find $\boldsymbol{E}'$ in this case. Assume that $eEa_B \ll \hbar/\tau$, where the natural radiative lifetime of the 2p state is given by

$$\frac{1}{\tau} = \frac{8\pi^2e^2\omega_{eg}^2a_B^2z_{eg}^2}{3hc^3}. \tag{3.5.64}$$

The matrix element of electron displacement along the z axis is

$$z_{eg} = \int \phi_e^* z\phi_g\,d^3r, \tag{3.5.65}$$

and e is the elementary charge.

Since $|\psi\rangle$ is an eigenstate of the perturbed Hamiltonian, we have

$$0 = \frac{d}{dt}\langle\psi|\boldsymbol{p}|\psi\rangle = -\langle\psi|\nabla V|\psi\rangle = -\left\langle\psi\left|\left(ea_B\boldsymbol{E} - \frac{e^2\boldsymbol{r}}{a_B r^3}\right)\right|\psi\right\rangle, \tag{3.5.66}$$

and the result follows.

Since the electric field is directed along the z axis, it will only couple the 1s state to the 2p state with $m = 0$. The matrix element is

$$\langle 2p, 0|V(t)|1s\rangle = eEa_B z_{eg}\cos\omega_{eg}t. \tag{3.5.67}$$

Therefore, we have

$$\dot{c}_{2p}(t) = \frac{ieEa_B z_{eg}}{\hbar}e^{i\omega_{eg}t}\cos\omega_{eg}t. \tag{3.5.68}$$

However, this does not take into account the radiative decay of the 2p state. We are given that the radiative lifetime τ is much less than the timescale for excitation by the electric field. We can account for this phenomenologically by adding a decay term to the differential equation:

$$\dot{c}_{2p}(t) = -\frac{c_{2p}}{\tau} - \frac{ieEa_B z_{eg}}{\hbar}e^{i\omega_{eg}t}\cos\omega_{eg}t. \tag{3.5.69}$$

In the time average we can replace $e^{i\omega_{eg}t}\cos\omega_{eg}t$ by $\frac{1}{2}$, and so at steady state we have

$$c_{2p} = -\frac{ieEa_B z_{eg}\tau}{2\hbar}. \tag{3.5.70}$$

The induced field is then given by

$$\begin{aligned}\boldsymbol{E'} &= \frac{2e}{a_B^2}|c_{2p}|\langle 2p|\boldsymbol{r}/r^3|1s\rangle \\ &= \frac{e^2Ez_{eg}\tau}{4\sqrt{2}\pi\hbar a_B}\hat{\boldsymbol{z}}\int(r\cos\theta)(r\cos\theta/r^3)e^{-3r/2}\,d^3\boldsymbol{r} \\ &= \frac{4\sqrt{2}}{27}\frac{e^2z_{eg}\tau}{\hbar a_B}\boldsymbol{E}.\end{aligned} \tag{3.5.71}$$

3.6 Scattering Theory

When two particles love each other very much, sometimes they...have an interaction, so to speak. This is just a special case of time-dependent perturbation theory: an incoming particle approaches another particle, feels the interaction potential as it passes by, and possibly transitions into a different state.

However, it's somewhat annoying to think about the situation this way, because describing a localized particle moving through space requires the use of a wavepacket. Instead, we can take a time-independent stance. Starting with the free particle Hamiltonian H_0, which has eigenstates $\psi(\boldsymbol{r}) = e^{i\boldsymbol{k}\cdot\boldsymbol{r}}$, what happens when we add some interaction potential $V(r)$ near the origin? The state will change in some complicated way near the origin, but we generally don't care: all we measure

is the amplitude of the state at large distances from the scattering source. At large distances $V(r) = 0$, so we should still have an eigenfunction of the free-particle operator H_0. From symmetry and realistic physical considerations, we expect the state to look like

$$\psi(\boldsymbol{r}) = e^{i\boldsymbol{k}\cdot\boldsymbol{r}} + f(\theta,\phi)\frac{e^{ikr}}{r}. \tag{3.6.1}$$

That is, we have the original state plus a spherical wave emanating from the interaction source. The function $f(\theta,\phi)$ should depend in some way on $V(r)$, and should tell us something about the differential cross section.

In fact, we have

$$\frac{d\sigma}{d\Omega} = |f(\theta,\phi)|^2. \tag{3.6.2}$$

To prove this, we look at the probability current for the state $\psi(r)$. For any wavefunction $\psi(\boldsymbol{r})$, we have

$$\frac{\partial}{\partial t}\left(|\psi(\boldsymbol{r})|^2\right) = \psi^*(\boldsymbol{r})\frac{H}{i\hbar}\psi(\boldsymbol{r}) - \psi(\boldsymbol{r})\frac{H}{i\hbar}\psi^*(\boldsymbol{r}) = \frac{\hbar}{2mi}\left(-\psi^*\nabla^2\psi + \psi\nabla^2\psi^*\right). \tag{3.6.3}$$

Comparing this to the continuity equation $\frac{\partial\rho}{\partial t} + \nabla\cdot\boldsymbol{j} = 0$, we find a probability current of

$$\boldsymbol{j} = \frac{\hbar}{2mi}\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right). \tag{3.6.4}$$

Now substituting the scattered part of (3.6.1), the radial component is

$$j_r = \frac{\hbar k}{mr^2}|f(\theta,\phi)|^2. \tag{3.6.5}$$

We can express the current as a particle flux, i.e.,

$$j_r = \frac{(\text{velocity})\times(\text{particles per unit area})\times(d\sigma)}{r^2(d\Omega)} = \frac{\hbar k}{mr^2}\frac{d\sigma}{d\Omega}, \tag{3.6.6}$$

which proves (3.6.2).

Now that we have established the importance of $f(\theta,\phi)$, we should compute it. By conservation of energy, we know that the incoming state with momentum $\boldsymbol{k}$ will mix with other states with the same energy $\frac{\hbar^2k^2}{2m}$. So, we need to do degenerate perturbation theory in a case where the degenerate eigenspace is infinite-dimensional (spanned by all $\boldsymbol{k'}$ with $k'^2 = k^2$). This will look somewhat like the time-independent perturbation theory we've done before, except with integrals instead of sums. We start with the Schrödinger equation,

$$(H_0 + V)\,|\psi\rangle = E\,|\psi\rangle\,. \tag{3.6.7}$$

Substituting (3.6.1), we find

$$V\,|\psi\rangle = (E - H_0)\,|\psi_S\rangle\,, \tag{3.6.8}$$

where $|\psi_S\rangle$ is the scattered wave (the second term in (3.6.1)). It sure would be nice if we could invert $E - H_0$ to express $|\psi_S\rangle$ in terms of $|\psi\rangle$, but this operator is singular – this is the difficulty with degenerate perturbation theory. Pretend for a second that this isn't a problem, and let $G(E)(E - H_0) = 1$. Then we would have

$$|\psi_S\rangle = G(E)V\,|\psi\rangle\,, \tag{3.6.9}$$

or

$$|\psi\rangle = |\psi_i\rangle + G(E)V\,|\psi\rangle\,, \tag{3.6.10}$$

where $|\psi_i\rangle$ is the incoming wave. This is known as the Lippmann-Schwinger equation.

This is not a user-friendly equation, so let's work on it a bit. First, we need to figure out how to define $G(E)$ and what it is. Since the Hamiltonian has only real eigenvalues, $G(E + i\epsilon)$ is well-defined, so let

$$G_\pm(E) = \lim_{\epsilon\to 0} G(E \pm i\epsilon). \tag{3.6.11}$$

There are different analytic properties depending on which sign we take, so we'll have to figure out the sign later on (but spoiler alert, we'll end up using the positive sign).

Now we evaluate $G_\pm(z)$ for $\operatorname{Im} z \neq 0$, where there are no issues in defining it. In the momentum basis, $z - H_0$ is diagonal with

$$\langle \boldsymbol{k'}|(z - H_0)|\boldsymbol{k}\rangle = \left(z - \frac{\hbar^2 k^2}{2m}\right)\delta(\boldsymbol{k} - \boldsymbol{k'}). \tag{3.6.12}$$

Letting $E_k = \frac{\hbar^2 k^2}{2m}$, we thus find

$$G(z) = \int d^3\boldsymbol{k}\frac{|k\rangle\,\langle k|}{z - E_k}. \tag{3.6.13}$$

For the purpose of actual computations, we would like to know this operator in the position basis. The matrix elements are

$$\langle \boldsymbol{r'}|G(z)|\boldsymbol{r}\rangle = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\frac{e^{i\boldsymbol{k}\cdot(\boldsymbol{r'}-\boldsymbol{r})}}{z - E_k}. \tag{3.6.14}$$

Now, let $z = \frac{\hbar^2(q\pm i\epsilon)^2}{2m}$. We then have

$$\langle \boldsymbol{r'}|G_\pm(\hbar^2 q^2/2m)|\boldsymbol{r}\rangle = \frac{2m}{\hbar^2}\int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\frac{e^{i\boldsymbol{k}\cdot(\boldsymbol{r'}-\boldsymbol{r})}}{(q \pm i\epsilon)^2 - k^2} \tag{3.6.15}$$

$$= \frac{m}{2\pi^2\hbar^2}\int_0^\infty k^2\,dk\int_{-1}^1 d(cos\theta)\,\frac{e^{ik|\boldsymbol{r'}-\boldsymbol{r}|\cos\theta}}{(q \pm i\epsilon)^2 - k^2} \tag{3.6.16}$$

$$= \frac{m}{\pi^2\hbar^2|\boldsymbol{r'}-\boldsymbol{r}|}\int_0^\infty \frac{k\sin(k|\boldsymbol{r'}-\boldsymbol{r}|)\,dk}{(q \pm i\epsilon)^2 - k^2}. \tag{3.6.17}$$

Kind of gross, but okay. We can do this last integral using Cauchy's formula. The integrand is even, so we can extend the integral to the whole real line with a factor of $\frac{1}{2}$. Furthermore, since the integrand is real (well, almost), we can express the sin as an imaginary part, so

$$\langle \boldsymbol{r'}|G_\pm(\hbar^2 q^2/2m)|\boldsymbol{r}\rangle = \frac{m}{2\pi^2\hbar^2|\boldsymbol{r'}-\boldsymbol{r}|}\operatorname{Im}\int_{-\infty}^\infty \frac{ke^{ik|\boldsymbol{r'}-\boldsymbol{r}|}\,dk}{(q \pm i\epsilon)^2 - k^2}. \tag{3.6.18}$$

We can close this integral in the upper half plane, since the exponential in the numerator will suppress the contribution from the semicircle. If we choose the positive sign, the pole at $q + i\epsilon$ contributes; if we take the negative sign, the pole at $-q+i\epsilon$ contributes. In order to get an outgoing spherical wave, we make the former choice, and so we have

$$\langle \boldsymbol{r'}|G_+(\hbar^2 q^2/2m)|\boldsymbol{r}\rangle = \frac{m}{2\pi^2\hbar^2|\boldsymbol{r'}-\boldsymbol{r}|}\operatorname{Im}\left(-2\pi i\frac{qe^{iq|\boldsymbol{r'}-\boldsymbol{r}|}}{2q}\right) = -\frac{m}{2\pi\hbar^2|\boldsymbol{r'}-\boldsymbol{r}|}e^{iq|\boldsymbol{r'}-\boldsymbol{r}|}. \tag{3.6.19}$$

Lovely. Now let's use this in the Lippmann-Schwinger equation. All this legwork in finding the position-space matrix elements means we can take the inner product with $\langle \boldsymbol{r}|$ and find

$$\psi(\boldsymbol{r}) = e^{i\boldsymbol{k}\cdot\boldsymbol{r}} - \frac{m}{2\pi\hbar^2}\int d^3\boldsymbol{r'}\frac{e^{ik|\boldsymbol{r'}-\boldsymbol{r}|}}{|\boldsymbol{r'}-\boldsymbol{r}|}V(\boldsymbol{r'})\psi(\boldsymbol{r'}). \tag{3.6.20}$$

This is starting to look more user-friendly. We can do a little bit better by remembering that we care about the wavefunction far from the origin, while the potential is localized near the origin, so $|\boldsymbol{r'}-\boldsymbol{r}|$ will be very large. We can then approximate $|\boldsymbol{r}-\boldsymbol{r'}| \approx r - \hat{\boldsymbol{r}}\cdot\boldsymbol{r'}$, and so the integral becomes

$$\psi(\boldsymbol{r}) = e^{i\boldsymbol{k}\cdot\boldsymbol{r}} - \frac{m}{2\pi\hbar^2}\frac{e^{ikr}}{r}\int d^3\boldsymbol{r'}e^{-ik\hat{\boldsymbol{r}}\cdot\boldsymbol{r'}}V(\boldsymbol{r'})\psi(\boldsymbol{r'}). \tag{3.6.21}$$

This is exactly the form we sought, with

$$f(\theta,\phi) = -\frac{m}{2\pi\hbar^2}\int d^3\boldsymbol{r'}e^{-ik\hat{\boldsymbol{r}}\cdot\boldsymbol{r'}}V(\boldsymbol{r'})\psi(\boldsymbol{r'}), \tag{3.6.22}$$

where the dependence on θ and ϕ enters through $\hat{\boldsymbol{r}}$.

There is one last major issue with user-friendliness: we don't know $\psi(\boldsymbol{r'})$ *a priori*. In general this can make things very difficult, but sometimes we can approximate the hardness away. For example, if the interaction is weak, then $\psi(\boldsymbol{r'})$ will be very nearly its unscattered value $e^{i\boldsymbol{k}\cdot\boldsymbol{r}}$, and so we can substitute this into the integral to determine $f(\theta,\phi)$ to lowest order. This gives

$$f(\theta,\phi) = -\frac{m}{2\pi\hbar^2}\int d^3\boldsymbol{r'}\,e^{-i\boldsymbol{q}\cdot\boldsymbol{r'}}V(\boldsymbol{r'}), \tag{3.6.23}$$

where $\boldsymbol{q} = k\hat{\boldsymbol{r}} - \boldsymbol{k}$ is the change in momentum. This is known as the Born approximation.

Problem 3.26 (M98Q2)a) Calculate the differential cross-section, $d\sigma/d\Omega$, for a particle with mass m in the spherical potential $V(r) = V_0 e^{-(r/a)^2}$, in first-order Born approximation. You may need

$$\int_0^\infty \sin(r)e^{-(r/b)^2}r\,dr = \frac{\sqrt{\pi}}{4}b^3e^{-b^2/4}. \tag{3.6.24}$$

b) Calculate the total cross-section. It may be helpful to use the representation $|\boldsymbol{k}-\boldsymbol{k'}| = 2|\boldsymbol{k}|\sin(\theta/2)$, where θ is the angle between $\boldsymbol{k}$ and $\boldsymbol{k'}$.

c) For which values of V_0, a and/or k is the first-order Born approximation applicable?

First we use the Born approximation to find $f(\theta,\phi)$. We have

$$\begin{aligned} f(\theta,\phi) &= -\frac{mV_0}{2\pi\hbar^2}\int d^3\boldsymbol{r'}\,e^{-i\boldsymbol{q}\cdot\boldsymbol{r'}}e^{-(r'/a)^2} \\ &= -\frac{mV_0}{\hbar^2}\int_0^\infty r^2\,dr\int_{-1}^1 d(\cos\theta)\,e^{-iqr\cos\theta}e^{-(r/a)^2} \\ &= -\frac{2mV_0}{q\hbar^2}\int_0^\infty r\sin(qr)e^{-(r/a)^2}\,dr \\ &= -\frac{\sqrt{\pi}}{2}\frac{mV_0a^3}{\hbar^2}e^{-(qa)^2/4}. \end{aligned} \tag{3.6.25}$$

The magnitude of the momentum transfer is $q = 2k \sin \frac{\theta}{2}$, and so we have

$$\frac{d\sigma}{d\Omega} = \frac{\pi}{4} \left(\frac{mV_0 a^3}{\hbar^2} \right)^2 \exp\left(-a^2 k^2 \sin^2 \frac{\theta}{2} \right). \tag{3.6.26}$$

The total cross section is

$$\begin{aligned} \sigma &= 2\pi \int_0^\pi \frac{d\sigma}{d\Omega} \sin\theta \, d\theta \\ &= \frac{\pi^2}{2} \left(\frac{mV_0 a^3}{\hbar^2} \right)^2 \int_0^\pi \sin\theta \exp\left(-\frac{1}{2} a^2 k^2 (1 - \cos\theta) \right) d\theta \\ &= \frac{\pi^2}{2} \left(\frac{mV_0 a^2}{k\hbar^2} \right)^2 \left(1 - e^{-a^2 k^2} \right). \end{aligned} \tag{3.6.27}$$

The Born approximation is applicable whenever $\frac{f(\theta,\phi)}{r}$ is small in the region of the interaction, which gives $\frac{mV_0 a^2}{\hbar^2} \ll 1$.

Problem 3.27 (J00Q1)
A particle of mass m and energy $\frac{\hbar^2 k^2}{2m}$ scatters in a central potential $V(r)$ which is everywhere positive and vanishes rapidly as $r \to \infty$. Let $d\sigma/d\Omega$ be the differential cross section as computed in the Born approximation. For precisely backwards scattering you are given

$$\left. \frac{d\sigma}{d\Omega} \right|_{\text{back}} = A \frac{\exp(-4\lambda k)}{k^2} \tag{3.6.28}$$

where A, λ are given parameters.

a) Calculate $d\sigma/d\Omega$ in the same approximation for arbitrary scattering angle.

b) Calculate $V(r)$.

The scattering cross section depends on k through $q = 2k \sin \frac{\theta}{2}$. For back scattering $q = 2k$, so we have

$$\frac{d\sigma}{d\Omega} = A \frac{4 \exp(-2\lambda q)}{q^2} = A \frac{\exp\left(-4\lambda k \sin \frac{\theta}{2}\right)}{k^2 \sin^2 \frac{\theta}{2}}. \tag{3.6.29}$$

It follows that

$$f(\theta, \phi) = \sqrt{A} \frac{2 \exp(-\lambda q)}{q}. \tag{3.6.30}$$

We also have

$$f(\theta, \phi) = -\frac{m}{2\pi\hbar^2} \int d^3\boldsymbol{r} \, e^{-i\boldsymbol{q}\cdot\boldsymbol{r}} V(\boldsymbol{r}). \tag{3.6.31}$$

Taking the inverse Fourier transform gives

$$\begin{aligned} V(\boldsymbol{r}) &= -\frac{4\pi\hbar^2\sqrt{A}}{m}\int \frac{d^3\boldsymbol{q}}{(2\pi)^3}e^{i\boldsymbol{q}\cdot\boldsymbol{r}}\frac{e^{-\lambda q}}{q} \\ &= -\frac{2\hbar^2\sqrt{A}}{\pi m r}\int_0^\infty \sin(qr)e^{-\lambda q}\,dq \\ &= -\frac{2\hbar^2\sqrt{A}}{\pi m}\frac{1}{r^2+\lambda^2}. \end{aligned} \tag{3.6.32}$$

Problem 3.28 (M04Q3)
A beam of particles of mass m and energy E propagates along the z axis of a coordinate system, and scatters from the cubic potential

$$V = \begin{cases} v & \text{if } |x|<L, |y|<L, |z|<L, \\ 0 & \text{otherwise} \end{cases}. \tag{3.6.33}$$

where v is a small constant energy.

a) Use the Born approximation to find an explicit formula for the scattering cross section $\frac{d\sigma}{d\Omega}$ as a function of the angles θ and ϕ. Recall that spherical coordinates of a point in space (r,θ,ϕ) are related to Cartesian coordinates (x,y,z) by $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$ and $z = r\cos\theta$. The Born approximation is easy to evaluate in one coordinate system and hard in the other.

b) Under what circumstances is this approximation for the scattering cross section valid? Explain.

The Born approximation gives

$$\begin{aligned} f(\theta,\phi) &= -\frac{m}{2\pi\hbar^2}\int d^3\boldsymbol{r}\, e^{i\boldsymbol{q}\cdot\boldsymbol{r}}V(\boldsymbol{r}) \\ &= -\frac{mv}{2\pi\hbar^2}\left(\int_{-L}^{L} dx\, e^{iq_x x}\right)\left(\int_{-L}^{L} dy\, e^{iq_y y}\right)\left(\int_{-L}^{L} dz\, e^{iq_z z}\right) \\ &= -\frac{4mvL^3}{\pi\hbar^2}\operatorname{sinc}(q_xL)\operatorname{sinc}(q_yL)\operatorname{sinc}(q_zL). \end{aligned} \tag{3.6.34}$$

The incoming wavevector is $\frac{\sqrt{2mE}}{\hbar}\hat{\boldsymbol{z}}$, and so this becomes

$$f(\theta,\phi) = -\frac{4mvL^3}{\pi\hbar^2}\operatorname{sinc}\left(\frac{\sqrt{2mE}}{\hbar}L\sin\theta\cos\phi\right)\operatorname{sinc}\left(\frac{\sqrt{2mE}}{\hbar}L\sin\theta\sin\phi\right)\operatorname{sinc}\left(\frac{\sqrt{2mE}}{\hbar}L(\cos\theta-1)\right). \tag{3.6.35}$$

The differential cross section is

$$\frac{d\sigma}{d\Omega} = \left(\frac{4mvL^3}{\pi\hbar^2}\right)^2 \operatorname{sinc}^2\left(\frac{\sqrt{2mE}}{\hbar}L\sin\theta\cos\phi\right)\operatorname{sinc}^2\left(\frac{\sqrt{2mE}}{\hbar}L\sin\theta\sin\phi\right)\operatorname{sinc}^2\left(\frac{\sqrt{2mE}}{\hbar}L(\cos\theta-1)\right). \tag{3.6.36}$$

This approximation is valid whenever $f(\theta,\phi)$ is small compared to the length scale of the potential, i.e., when $\frac{mvL^2}{\hbar^2} \ll 1$.

In the last problem, there was some weirdass angle dependence because the potential wasn't spherically symmetric. Conversely, when the potential is spherically symmetric (the typical case), the weirdass-osity (technical term) is restricted by the symmetry. We can show this more clearly by using a basis $|k,\ell,m\rangle$ adapted to the conserved L^2 and L_z operators. From (3.3.43), the wavefunction of this state satisfies

$$u'' - \left(\frac{\ell(\ell+1)}{r^2} + \frac{2mV(r)}{\hbar^2} - k^2\right)u = 0. \tag{3.6.37}$$

As $r\to\infty$, the potential vanishes and we can ignore the centrifugal term, and the equation becomes $u''+k^2u=0$, which implies the wavefunction is given by

$$\psi_{k,\ell,m}(r,\theta,\phi) = \left(A\frac{e^{ikr}}{r} + B\frac{e^{-ikr}}{r}\right)Y_\ell^m(\theta,\phi). \tag{3.6.38}$$

Conservation of probability requires $|A|=|B|$, so we can express the radial part as $\sin(kr-\varphi_\ell)$. A detailed solution to the free-particle radial equation (including the centrifugal term) gives

$$\psi_{k,\ell,m}(r,\theta,\phi) = \sqrt{\frac{2}{\pi}}kj_\ell(kr)Y_\ell^m(\theta,\phi). \tag{3.6.39}$$

The functions $j_\ell(kr)$ are spherical Bessel functions (but fun fact, they can be expressed in terms of elementary functions). Their asymptotic behavior allows us to fix $\varphi_\ell^{(0)} = \frac{\ell\pi}{2}$. When we include the potential, we can parametrize its effects in terms of phase shifts $\varphi_\ell = \varphi_\ell^{(0)} - \delta_\ell$.

Now we expand the scattered state in these bases. For the free particle basis, this amounts to finding an expansion of $f(\theta,\phi)$ in spherical harmonics. Spherical symmetry means there can be no ϕ dependence, so in fact we just have $f(\theta)$ expanded in the $m=0$ components:

$$f(\theta) = \sum_\ell f_\ell Y_\ell^0(\theta). \tag{3.6.40}$$

Then we can expand the incoming wave as

$$e^{ikz} = \sum_\ell B_\ell j_\ell(kr)Y_\ell^0(\theta), \tag{3.6.41}$$

so that the total scattered state is

$$\sum_\ell \left(B_\ell j_\ell(kr) + \frac{f_\ell e^{ikr}}{r}\right)Y_\ell^0(\theta). \tag{3.6.42}$$

Alternatively, we could expand in the basis with phase shifts, giving

$$\sum_\ell A_\ell \frac{\phi_{k,\ell}(r)}{r}Y_\ell^0(\theta). \tag{3.6.43}$$

Equating these two, we find

$$B_\ell j_\ell(kr) + \frac{f_\ell e^{ikr}}{r} = A_\ell \frac{\phi_{k,\ell}(r)}{r}, \tag{3.6.44}$$

which implies the asymptotic relation

$$B_\ell \sin(kr - \ell\pi/2) + f_\ell e^{ikr} = A_\ell \sin(kr - \ell\pi/2 + \delta_\ell). \tag{3.6.45}$$

The coefficients B_ℓ do not depend on the potential, and in fact we have $B_\ell = i^\ell\sqrt{4\pi(2\ell+1)}$. Expanding everything in terms of $e^{\pm ikr}$ and solving for f_ℓ gives

$$f_\ell = \sqrt{4\pi(2\ell+1)}\frac{e^{i\delta_\ell}\sin\delta_\ell}{k}. \tag{3.6.46}$$

It follows that

$$\frac{d\sigma}{d\Omega} = \frac{4\pi}{k^2}\left|\sum_{\ell=0}^{\infty}\sqrt{2\ell+1}f_\ell e^{i\delta_\ell}\sin\delta_\ell Y_\ell^0(\theta)\right|^2. \tag{3.6.47}$$

By orthogonality of the spherical harmonics, the total cross section reduces to

$$\sigma = \frac{4\pi}{k^2}\sum_{\ell=0}^{\infty}(2\ell+1)\sin^2\delta_\ell. \tag{3.6.48}$$

So, we've reduced the whole problem to computing some pesky phase shifts.

Problem 3.29 (M05Q2)
A particle of mass m and kinetic energy E scatters from a thin spherical shell of radius R. The scattering potential can be approximated by

$$V = vR\delta(r-R) \tag{3.6.49}$$

where r is the distance of the particle from the scattering center, v is a characteristic energy, and δ denotes a Dirac delta function.

a) Derive the S-wave scattering cross-section σ_0 in terms of E, m, v, R, and $\hbar$.

b) For what energies E does σ_0 vanish? Explain.

c) Derive a formula for σ_0 when $E = 0$.

d) For what values of v does $\sigma_0 \to \infty$ as $E \to 0$ in c)? Explain.

The radial Schrödinger equation is

$$\frac{d^2u}{dr^2} + \left(\frac{2mE}{\hbar^2} - \frac{2mvR}{\hbar^2}\delta(r-R)\right)u = 0. \tag{3.6.50}$$

Let $k = \frac{\sqrt{2mE}}{\hbar}$. Since $u(0) = 0$, we have $u(r) \sim \sin kr$ for $r < R$. For $r > R$, we have $u(r) \sim A\sin(kr + \delta_0)$. To solve for δ_0, we use the boundary condition at R,

$$u'(R+0) - u'(R-0) = \frac{2mvR}{\hbar^2}u(R), \qquad u(R+0) = u(R-0). \tag{3.6.51}$$

This implies

$$Ak\cos(kR+\delta_0) - k\cos(kR) = \frac{2mvR}{\hbar^2}\sin(kR), \tag{3.6.52}$$

$$A\sin(kR+\delta_0) = \sin(kR). \tag{3.6.53}$$

Dividing these and solving, we find

$$\delta_0 = \cot^{-1}\left(\frac{2mvR}{\hbar^2 k} - \cot(kR)\right) - kR, \tag{3.6.54}$$

which gives

$$\sigma_0 = \frac{4\pi}{k^2}\sin^2\delta_0 = \frac{4\pi}{k^2}\left(\frac{2mvR}{\hbar^2}\right)^2 \frac{\sin^2(kR)}{k^2 + (2mvR/\hbar^2 - k\cot kR)^2}. \tag{3.6.55}$$

From this we see that σ_0 vanishes when $kR = \pi n$, or when $E = \frac{n^2\pi^2\hbar^2}{2mR^2}$.

Taking the limit as $k \to 0$, we find

$$\sigma_0(E=0) = 4\pi R^2\left(1 - \frac{\hbar^2}{2mvR^2}\right)^{-2}. \tag{3.6.56}$$

When $v = \frac{\hbar^2}{2mR^2}$, we have $\sigma_0 \to \infty$ as $E \to 0$. This is the value of v for which there is a bound state at $E = 0$.

What about computing phase shifts in more general cases? We go back to the Schrödinger equation. Assume the potential vanishes outside some radius R. Then the phase-shifted asymptotic solution corresponds to the following exact radial wavefunction for $r > R$:

$$R_\ell(r) = e^{i\delta_\ell}\left(\cos\delta_\ell j_\ell(kr) - \sin\delta_\ell n_\ell(kr)\right), \tag{3.6.57}$$

where j_ℓ is a spherical Bessel function as before and n_ℓ is a spherical Bessel function of the second kind. Note that

$$j_0(x) = \frac{\sin x}{x}, \qquad n_0(x) = -\frac{\cos x}{x}. \tag{3.6.58}$$

Using this, we can evaluate a logarithmic derivative at $r = R$:

$$\beta_\ell = \left(\frac{r}{R_\ell}\frac{dR_\ell}{dr}\right)_{r=R} = kR\frac{j'_\ell(kR)\cos\delta_\ell - n'_\ell(kR)\sin\delta_\ell}{j_\ell(kR)\cos\delta_\ell - n_\ell(kR)\sin\delta_\ell}. \tag{3.6.59}$$

Solving this for δ_ℓ, we find

$$\tan\delta_\ell = \frac{kRj'_\ell(kR) - \beta_\ell j_\ell(kR)}{kRn'_\ell(kR) - \beta_\ell n_\ell(kR)}. \tag{3.6.60}$$

Computing phase shifts is thus reduced to computing the β_ℓ. We can find these by solving the radial Schrödinger equation for $r < R$, where the potential is nonzero.

Problem 3.30 (J03Q1)
Consider the scattering of quantum-mechanical particles by a spherical square-well in three

dimensions given by a radial potential

$$V(r) = \begin{cases} V_0 & \text{for } r < a \\ 0 & \text{otherwise} \end{cases} \tag{3.6.61}$$

having a constant depth V_0 within a radius $a > 0$ of the origin. Assume that the particles have an extremely low energy $E > 0$, that is, $a\sqrt{2mE} \ll \hbar$. In this case only partial waves of angular momentum $L = 0$ suffer appreciable scattering.

a) Calculate the total cross section for the case of an attractive potential with depth $V_0 < 0$.

b) Starting from the answer you derived, consider now the case of scattering from a hard sphere, by taking the potential to be repulsive ($V_0 > 0$) in the limit $V_0/E \to \infty$. Show that the answer is $4\pi a^2$ (four times bigger than the classical result).

We are assuming only the $\ell = 0$ mode contributes, and so the total cross section is just $\sigma = \frac{2\pi\hbar^2}{mE}\sin^2\delta_0$. To find δ_0, we use the Schrödinger equation for $r < R$ and $\ell = 0$,

$$\frac{d^2u}{dr^2} + \frac{2m(E - V_0)}{\hbar^2}u = 0. \tag{3.6.62}$$

Since $u(0) = 0$, the solution is $u(r) \sim \sin\left(\frac{\sqrt{2m(E-V_0)}}{\hbar}r\right)$, giving

$$\beta_0 = x\cot x, \tag{3.6.63}$$

where $x = a\frac{\sqrt{2m(E-V_0)}}{\hbar}$. Letting $y = a\frac{\sqrt{2mE}}{\hbar}$, this gives

$$\begin{aligned} \tan\delta_0 &= \frac{yj_0'(y) - x\cot x j_0(y)}{yn_0'(y) - x\cot x n_0(y)} \\ &= \frac{y\cos y - \sin y - x\cot x\sin y}{y\sin y + \cos y + x\cot x\cos y}. \end{aligned} \tag{3.6.64}$$

We are given $y \ll 1$, so making the appropriate approximations we find

$$\tan\delta_0 = \frac{-xy\cot x}{1 + x\cot x}. \tag{3.6.65}$$

This implies

$$\sigma = 4\pi a^2\frac{x^2\cot^2 x}{1 + 2x\cot x + x^2(1 + y^2)\cot^2 x}. \tag{3.6.66}$$

Now we take a repulsive potential with $V_0/E \to \infty$. This means $|x| \gg 1$, and so

$$\sigma \to \frac{4\pi a^2}{1 + y^2} \approx 4\pi a^2. \tag{3.6.67}$$

3.7 Additional Problems

Problem 3.31 (J13Q2)
Consider two indistinguishable nonrelativistic bosons of mass m, constrained to move one-dimensionally around a circle of perimeter L. The particles each have spin 1, and they interact via a spin-independent potential that is a Dirac delta function: $V(x_1, x_2) = g\delta(x_1 - x_2)$, where x_i is the position on the circle (in arc length) of particle i.

a) First look at zero interaction, $g = 0$, being careful to only include states of the correct symmetry for these indistinguishable spin-1 bosons. What are the energies and the degeneracies of the ground state and of the lowest energy excited state? In each case, say what value(s) of total spin these states may have.

b) Add a weak interaction $g \neq 0$. Now what are the degeneracies of the ground state and the lowest-energy excited state? For each sign of g, say what value(s) of total spin these states may have.

c) Solve for a two particle ground state wavefunction, including showing the spin state. Do this first at $g = 0$, and then at all other $g \neq 0$. In the latter case you may leave one parameter in the wavefunction specified only as the solution to an equation that you will not be able to solve analytically.

The spatial states for a single particle are given by the wavefunction

$$\psi_n(x) \sim e^{2\pi i n x/L}, \tag{3.7.1}$$

where $n \in \mathbb{Z}$. The energy of a state is $\frac{\hbar^2}{2mL^2}n^2$. In each spatial state the particle can have spin component -1, 0, or 1.

To form the two-particle state, the spatial and spin wavefunctions must be either both symmetric or both antisymmetric. The ground state will be the symmetric combination of ψ_0 with itself, and spins in a symmetric combination; the spin can be 0 or 2, giving a total degeneracy of 6. The lowest excited state will be a combination of ψ_0 and $\psi_{\pm 1}$; the combination can be symmetric, in which case the spin is 0 or 2, or antisymmetric, in which case the spin is 1. The total degeneracy is 18.

The degeneracy of the ground state is due to spin alone, so when we add the spin-independent interaction the ground state degeneracy is unchanged. The degeneracy between the symmetric and antisymmetric combinations of the spatial ψ_0 and $\psi_{\pm 1}$ wavefunctions is broken by the potential. For $g > 0$, the first excited state is the antisymmetric combination, which has total spin 1, and total degeneracy 6. For $g < 0$, the first excited state is the symmetric combination, with total spin 0 or 2 and total degeneracy 12.

To solve for the ground state wavefunction, we start with general solutions to the free-particle Schrödinger equation for both $x_1 < x_2$ and $x_1 > x_2$:

$$\psi(x_1, x_2) = \begin{cases} A\exp(i(k_1x_1 + k_2x_2)) + B\exp(i(k_2x_1 + k_1x_2)) & x_1 < x_2, \\ C\exp(i(k_1x_1 + k_2x_2)) + D\exp(i(k_2x_1 + k_1x_2)) & x_1 > x_2 \end{cases}. \tag{3.7.2}$$

The ground state will be symmetric, and $\psi(x_1, x_2) = \psi(x_2, x_1)$ forces $A = D$, $B = C$. To relate A and B, we use the boundary condition at the delta function. Integrating the Schrödinger equation on a small line from $(x - \epsilon, x)$ to $(x + \epsilon)$ gives

$$-\frac{\hbar^2}{2m}\Delta\left(\frac{\partial\psi}{\partial x_1}\right) + g\psi = 0. \tag{3.7.3}$$

Substituting our ansatz gives

$$i(k_2 - k_1)(A - B) = g(A + B). \tag{3.7.4}$$

We can solve this for B/A, giving

$$\frac{B}{A} = -\frac{g + i(k_1 - k_2)}{g - i(k_1 - k_2)} \equiv e^{i\alpha}. \tag{3.7.5}$$

Up to normalization this fixes all the coefficients. To fix k_1 and k_2 we use periodicity; periodicity between $(x_1, x_2) = (0, 0)$ and $(L, 0)$ requires

$$e^{ik_1 L} + e^{i(k_2 L + \alpha)} = 1 + e^{i\alpha}. \tag{3.7.6}$$

Similarly, for $(-L, 0)$ we obtain

$$e^{-ik_2 L} + e^{i(-k_1 L + \alpha)} = 1 + e^{i\alpha}. \tag{3.7.7}$$

Clearly we should have $k_1 = -k_2 \equiv k$. Taking norms gives

$$\cos(2kL - \alpha) = \cos\alpha, \tag{3.7.8}$$

so $kL = \alpha + \pi n$. For the ground state we want the lowest value of k^2, so we take $n = 0$. It follows that k will be the solution to

$$kL = \alpha = \pi + 2\tan^{-1}\frac{2k}{g}, \tag{3.7.9}$$

which we cannot solve analytically. In summary, the ground state spatial wavefunction is

$$\psi(x_1, x_2) = A\begin{cases}\exp(ik(x_1 - x_2)) + e^{i\alpha - ik(x_1 - x_2)} & x_1 < x_2, \\ \exp(-ik(x_1 - x_2)) + e^{i\alpha + ik(x_1 - x_2)} & x_2 > x_1\end{cases}, \tag{3.7.10}$$

and the spin state is one of the six symmetric states.

Problem 3.32 (M01Q2)
Consider two hydrogen atoms with a fixed distance r between their nuclei that is large compared to the size of the atoms. Treat the Coulomb interaction as instantaneous (no retardation), and neglect the interactions between the spins.

a) The ground state energy of this pair of atoms depends on r as $C_0 + A_0 r^{-\delta_0} + \ldots$, where C_0, A_0 are constants. Find δ_0.

b) Give an order of magnitude estimate for A_0 and give a general argument why A_0 should be negative.

c) Now consider the first excited state of the system (keeping the distance r between the nuclei fixed and large). The energy depends on r as $C_1 + A_1 r^{-\delta_1} + \ldots$. Find δ_1.

d) Estimate at what distance (between the atoms) you will have to take into account the retardation effects in electromagnetism.

Each atom will be slightly polarized in the same direction, so each atom experiences the dipole field of the other. The dipole moment of each atom is proportional to the dipole field it experiences. Since dipole fields fall off as r^{-3}, the interaction energy will fall off as r^{-6}, so $\delta_0 = 6$.

The dipole field will have magnitude $\frac{ea_0}{r^3}$. The dipole moment will be proportional to this, and by dimensional analysis the constant of proportionality should be of order a_0^3. This gives $A_0 \sim e^2 a_0^5$.

In the first excited state, one of the atoms is in the p state, so thinking classically it already has a dipole moment of order ea_0. This dipole moment induces a dipole in the other atom, which then feels the dipole field, and so we have $\delta_1 = 3$.

Retardation effects become important when an electron can "move" around its atom in the time it takes for electromagnetic effects to propagate back and forth. By the uncertainty principle the electron velocity is of order $\frac{\hbar}{m_e a_0}$, and so retardation is important when $r \sim \frac{m_e a_0 c}{\hbar} a_0$. Substituting the actual values of these quantities gives about $20a_0$.

Problem 3.33 (J04Q2)
Two observers in different inertial frames will need different wave functions to describe the same physical system. To make things simple we will consider how it works in one dimension: The first observer uses coordinates (x, t) and a wave function $\psi(x, t)$ while the second uses (x', t) and $\hat{\psi}(x', t)$ with, of course, $x' = x - vt$, v a constant velocity. The wave functions for the two observers are said to be related as follows:

$$\hat{\psi}(x', t) = \psi(x, t) \exp\left(-\frac{i}{\hbar}\left[mvx - \frac{m}{2}v^2 t\right]\right). \tag{3.7.11}$$

Despite its innocuous look (it's just a phase!) this transformation has interesting effects!

a) Let's verify that it makes sense. Suppose $\psi(x, t)$ is the wave function for a free particle of momentum p. Show that $\hat{\psi}(x', t)$ is the wave function of a free particle with a different momentum. What is its momentum?

b) Now let's put this to work. Suppose we have a harmonic oscillator in its ground state for < 0; its wave function is

$$\psi(x, t) = N \exp\left(-\frac{m\omega}{2\hbar}x^2 - \frac{i}{2}\omega t\right), \tag{3.7.12}$$

where N is a normalization constant. Suppose that at $t = 0$ the potential suddenly starts to move at velocity v. Because the wave function does not change immediately, it is no longer the ground state wave function of the moving harmonic oscillator. What is the probability of finding the system in the moving ground state at a later time $t > 0$?

If $\psi(x,t) = \exp\left(\frac{ipx}{\hbar} - i\frac{p^2 t}{2m\hbar}\right)$, then we have

$$\begin{aligned}\hat{\psi}(x',t) &= \exp\left(\frac{i(p-mv)(x'+vt)}{\hbar} - i\frac{p^2 t}{2m\hbar} + i\frac{mv^2 t}{2\hbar}\right) \\ &= \exp\left(\frac{i(p-mv)x'}{\hbar} - i\frac{(p-mv)^2 t}{2m\hbar}\right),\end{aligned} \tag{3.7.13}$$

as expected.

To solve the harmonic oscillator problem, we use the given transformation to view the system in a frame where the particle is suddenly given a velocity $-v$ at $t = 0$. At $t = 0$ the wavefunction will be

$$\psi(x, t=0) = N\exp\left(-\frac{m\omega}{2\hbar}x^2 - \frac{i}{\hbar}mvx\right). \tag{3.7.14}$$

Absorbing an additional factor into the normalization, this is

$$\psi(x, t=0) = N\exp\left(-\frac{m\omega}{2\hbar}\left(x + \frac{iv}{\omega}\right)^2\right). \tag{3.7.15}$$

Acting with the lowering operator $a = \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{\hbar}{m\omega}\frac{d}{dx}\right)$ gives

$$a\left|\psi\right\rangle = -i\sqrt{\frac{mv^2}{2\hbar\omega}}\left|\psi\right\rangle, \tag{3.7.16}$$

so we have a coherent state. Let $\alpha = -i\sqrt{\frac{mv^2}{2\hbar\omega}}$. Then we have

$$\left|\psi\right\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\left|n\right\rangle, \tag{3.7.17}$$

so the probability of finding the system in its ground state will be

$$P_0 = \exp\left(-\frac{mv^2}{2\hbar\omega}\right). \tag{3.7.18}$$

Chapter 4

Statistical Mechanics

True story: one day, Ludwig Boltzmann decided he was going to use the microscopic physics of molecules to understand the macroscopic behavior of gases. Naturally, he needed to tabulate the positions and velocities of every gas molecule so that he could use Newton's laws to evolve the system forward and see where it would lead. Boltzmann fired up MS Excel and started tabulating. Modern versions of Excel have about a million rows, but Boltzmann lived a long time ago, when Excel only had 65,536 rows. So, when Boltzmann got to the $65{,}537^{\text{th}}$ molecule, he had to scrap the project and develop statistical mechanics instead.

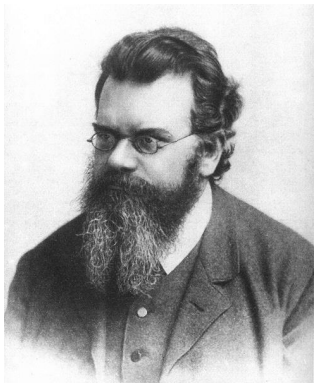
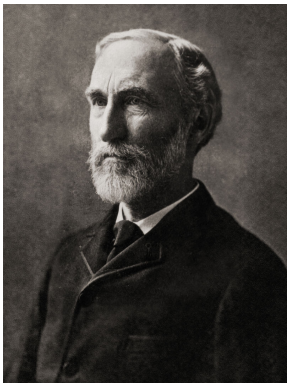




Figure 4.1: (Left) Sadi Carnot, whose bombass-titled book *Reflections on the Motive Power of Fire* kicked off the study of thermodynamics. (Center) Ludwig Boltzmann, spreadsheet wizard and discoverer of the statistical interpretation of entropy. (Right) Josiah Willard Gibbs, who initiated a policy making energy available on a government-run exchange. He was planning to just call it "free energy" but conservatives opposed the policy and branded it as "Gibbs free energy." The program's website had serious problems at its launch, but now it's working fine.

In 4.1, we'll review thermodynamics and heat engines and stuff. In 4.2 and 4.3, we'll look at statistical mechanics and partition functions and see how things match up with thermodynamics. In 4.4 we'll let particles be friends with each other. In 4.5 and 4.6, we'll mix in some Wacky Wednesday fanfiction.

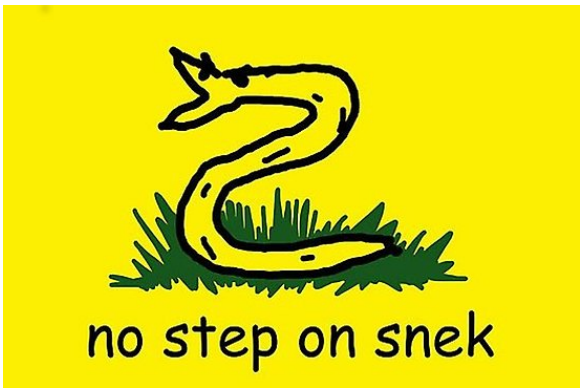


Figure 4.2: "No step on snek" ∼ Ben Franklin.

4.1 Classical Thermodynamics

When you step on a snek, snek no like it. Snek pushes back. No step on snek.

The same principle holds for gases. No step on gas. More precisely, if you step on a gas, it will push back with some pressure p. If p is greater than the pressure of the environment, the gas may expand by some volume ΔV, in which case the environment does work

$$W = -p\,\Delta V \tag{4.1.1}$$

on the gas.

We might be tempted to identify this work with the change in energy of the gas, much like we think of the work $W = kx\Delta x$ done on a spring as the change in its energy. However, pressure-volume work is not the only means by which we can change the energy of a gas. For example, what if Brad Pitt walks into the room? No pressure-volume work is done, but we all know there's more energy in that room. We refer to energy transfer via Brad Pitt injection or similar processes as heat. The total differential of the internal energy U is then given by

$$dU = đW + đQ = -p\,dV + đQ \tag{4.1.2}$$

where $đQ$ denotes heat transfer. The symbol $đ$ denotes an infinitesimal quantity which is not an exact differential; we say that U is a state function, whereas work and heat are not state functions. That's a little piece of $đ$-trivia.

From this expression alone we can obtain some observables. The heat capacity of a gas is the heat required to raise it by one unit of temperature; it can be defined for temperature increases at constant volume or at constant pressure. The heat capacities are

$$C_V = \left(\frac{đQ}{dT}\right)_V = \left(\frac{dU + p\,dV}{dT}\right)_V = \left(\frac{\partial U}{\partial T}\right)_V, \tag{4.1.3}$$

$$C_p = \left(\frac{đQ}{dT}\right)_p = \left(\frac{dU + p\,dV}{dT}\right)_p = \left(\frac{\partial U}{\partial T}\right)_p + p\frac{dV}{dT}. \tag{4.1.4}$$

Now seems like a good time to introduce the ideal gas: a gas with equation of state $PV = Nk_BT$, where N is the number of particles and k_B is the Boltzmann constant (which is just a unit conversion, it has no physical significance). This equation of state implies that U is a function of T

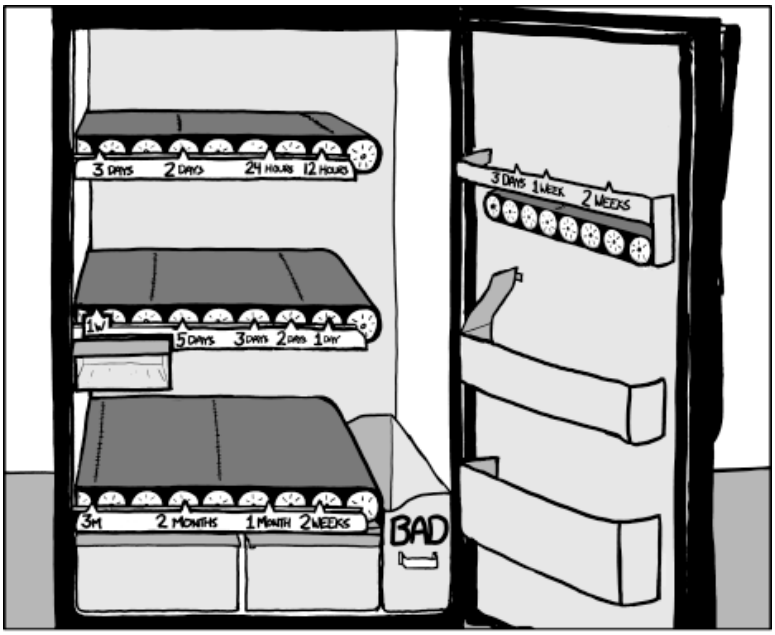


Figure 4.3: Ideal, but not perfect, because of the second law of thermodynamics.

alone, and so for an ideal gas we have

$$C_V = \frac{dU}{dT}, \qquad C_p = \frac{dU}{dT} + Nk_B. \tag{4.1.5}$$

Intuitively, $C_p > C_V$ because if volume is not held constant, the gas will expand as we add heat, thus dissipating some of the energy gain in the form of work done on the environment, so it takes more heat to get the same temperature increase.

Even though $đW$ is not an exact differential, we can express it in terms of the exact differential dV. Ideally we would like to do the same thing for the heat transfer $đQ$, but this will require some thought. We will base everything on the second law of thermodynamics, which says that it is impossible to build a perfect refrigerator – a device which transfers heat up a temperature gradient with no side effect.

Thermodynamics began with the study of heat engines, which are supposed to extract work from a temperature gradient. Sadi Carnot, in his badass book "On the Motive Power of Fire," showed how to construct the most efficient possible heat engine between hot and cold temperatures T_H and T_C. The idea is simple: extract heat Q_H from a reservoir at T_H, cool the engine to T_C in a way that does not involve heat transfer – "adiabatically" – give heat Q_C to the reservoir at T_C, and then heat the engine back to T_H adiabatically.

Since we come back to the same place we started, there can be no total change in the internal energy of the engine, since U is a state function. The engine receives a net height $Q_H - Q_C$, so it must provide exactly this much work to its environment to balance energy. We had to input energy Q_H (in the form of heat) to make the cycle go. Thus, the efficiency is

$$\eta = \frac{Q_H - Q_C}{Q_H} = 1 - \frac{Q_C}{Q_H}. \tag{4.1.6}$$

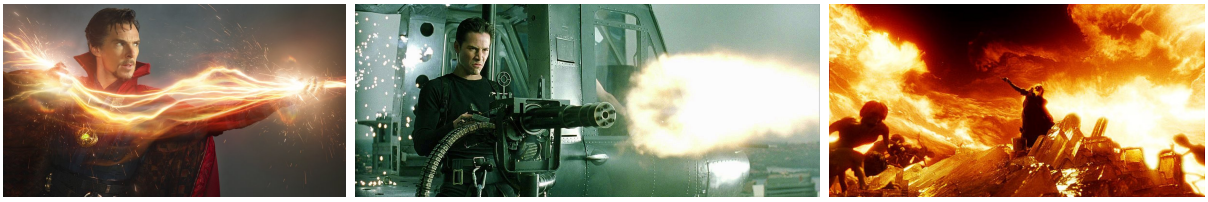


Figure 4.4: The many incarnations of Sadi Carnot, master of fire.

Imagine we could do better than this; a heat engine which extracts heat Q'_H and deposits Q'_C, so its efficiency is

$$\eta' = 1 - \frac{Q'_C}{Q'_H} > \eta. \tag{4.1.7}$$

Fancypants engine thinks it can beat Carnot? NOBODY BEATS CARNOT. Carnot is Benedict Cumberbatch, Keanu Reeves and Albus Fucking Dumbledore all rolled into one (see Figure 4.4).

Let's see why Carnot is an unbeatable sorceror-god. Let $Q'_H - Q'_C = \alpha(Q_H - Q_C)$. We can approximate α arbitrarily well by a rational $\frac{N}{N'}$. Then imagine running the hypothetical engine N' times, and running the Carnot engine N times backwards (a key property of the Carnot engine is its reversibility). This composite process would have the net effect of transporting heat $NQ_H - N'Q'_H$ from the cold reservoir to the hot reservoir, without doing any work. Uh oh, sounds like a perfect refrigerator – in order to avoid breaking the second law, we need $NQ_H - N'Q'_H < 0$, or $Q'_H > \frac{N}{N'}Q_H = \alpha Q_H$. But this means

$$\eta' = \frac{Q'_H - Q'_C}{Q'_H} = \frac{\alpha(Q_H - Q_C)}{Q'_H} < \frac{Q_H - Q_C}{Q_H} = \eta. \tag{4.1.8}$$

Don't mess with Carnot, motherfuckers.

The expression we have for the Carnot efficiency isn't particularly useful; it would be better to have η in terms of T_H and T_C directly. The copout answer is to say that we haven't yet defined temperature, so we might as well define it such that

$$\eta_{\text{Carnot}} = 1 - \frac{T_C}{T_H}. \tag{4.1.9}$$

We can motivate this a bit better by looking at the Carnot cycle for an ideal gas. In this case we instead define temperature to be proportional to the internal energy per particle, so the equation of state implies $U = \alpha pV$ for some dimensionless constant α (which depends on the gas; later we will show that $\alpha = 3/2$ for monatomic gases and $5/2$ for diatomic gases). The heat exchanges then take place along isotherms $pV = \text{const}$. The temperature changes take place along "adiabats," paths for which $đQ = 0$. This implies

$$0 = đQ = dU - đW = \alpha(p\,dV + V\,dp) + p\,dV = (\alpha+1)p\,dV + \alpha V\,dp. \tag{4.1.10}$$

This implies $pV^\gamma = \text{const}$ where $\gamma = \frac{\alpha+1}{\alpha}$. Now we can construct a Carnot cycle explicitly in $p-V$ space. Starting at (p_0, V_0), we move along an isotherm to $(\xi_1^{-1}p_0, \xi_1 V_0)$. We then follow an isotherm to $(\xi_1^{-1}\xi_2^{-\gamma}p_0, \xi_1\xi_2 V_0)$. We have to return to the original state by following an isotherm and then an adiabat; clearly the only way to do so is through the point $(\xi_2^{-\gamma}p_0, \xi_2 V_0)$.

To find the efficiency, we need the heat transfers Q_H and Q_C. Along the isotherm at T_H we have

$$đQ = (\alpha+1)p\,dV + \alpha V\,dp = p_0 V_0 \frac{d\xi}{\xi}, \tag{4.1.11}$$

and so $Q_H = p_0 V_0 \log \xi_1$. Similarly, $Q_C = \xi_2^{1-\gamma} p_0 V_0 \log \xi_1$, so

$$\eta = 1 - \xi_2^{1-\gamma}. \tag{4.1.12}$$

From the equation of state for the ideal gas, we clearly have $\frac{T_C}{T_H} = \xi_2^{1-\gamma}$, so in fact

$$\eta = 1 - \frac{T_C}{T_H}. \tag{4.1.13}$$

At last we can achieve our goal of rewriting $đQ$. We have shown that for a Carnot cycle, $\frac{Q_C}{Q_H} = \frac{T_C}{T_H}$. Put another way, the Carnot cycle satisfies

$$\oint \frac{đQ}{T} = 0. \tag{4.1.14}$$

In fact, this holds for any reversible cycle, since we can approximate any reversible cycle by many small Carnot cycles. Thus, for reversible cycles, $\frac{đQ}{T}$ is an exact differential. We define

$$dS = \frac{đQ}{T}, \tag{4.1.15}$$

and call S the entropy. Now we have

$$dU = T\,dS - p\,dV. \tag{4.1.16}$$

Additionally, the indomitability of Carnot implies that for a generic (not necessarily reversible) cycle, $\oint \frac{đQ}{T} \geq 0$, so the entropy S never decreases. This is the more familiar statement of the second law of thermodynamics, but it is equivalent to forbidding perfect refrigerators.

Problem 4.1 (M12T2)
We have two large solid blocks with heat capacities C_1 and C_2. Assume these heat capacities are each constant in the range of temperatures considered in this problem. Initially the two blocks are at temperatures T_{1o} and T_{2o} and have entropies S_{1o} and S_{2o}, respectively, with $T_{1o} > T_{2o}$. In this problem there are no volume changes.

a) Let these two blocks be in an isolated enclosure, so that no heat or work can flow in or out from the rest of the universe. What is the maximum total entropy that this system of blocks can reach? How do you describe this maximum entropy state? Justify your answer.

b) Alternatively, run a very small, reversible Carnot heat engine between the two blocks until equilibrium is reached and no more work can be extracted. What is the final temperature? How much work did the engine do?

Since the system is isolated, and there are no volume changes (so no pressure-volume work is done), the blocks can only interact by exchanging heat. When a quantity $đQ$ of heat is transferred from the first block to the second block, we have the following relationships:

$$dS = \frac{đQ}{T_2} - \frac{đQ}{T_1}, \qquad dT_1 = -\frac{đQ}{C_1}, \qquad dT_2 = \frac{đQ}{C_2}. \tag{4.1.17}$$

Thus, the maximum total entropy will be reached when $T_1 = T_2$, after which any further heat transfer would decrease entropy. Integrating the second two equations we find $T_1 = T_{1o} - \frac{\Delta Q}{C_1}$ and $T_2 = T_{2o} + \frac{\Delta Q}{C_2}$, and substituting these relations into the entropy equation we find

$$dS = \frac{đQ}{T_{2o} + \Delta Q/C_2} - \frac{đQ}{T_{1o} - \Delta Q/C_1}. \tag{4.1.18}$$

Integrating this, we find

$$\Delta S = \int_0^{\Delta Q} \left(\frac{dx}{T_{2o} + x/C_2} - \frac{dx}{T_{1o} - x/C_1} \right) = C_2 \log \left(1 + \frac{\Delta Q}{C_2 T_{2o}} \right) - C_1 \log \left(1 - \frac{\Delta Q}{C_1 T_{1o}} \right), \tag{4.1.19}$$

and so the maximum total entropy is $S_{1o} + S_{2o} + \Delta S$. From the equilibrium condition, we find that the total heat transfer is $\Delta Q = (T_{1o} - T_{2o}) \left(C_1^{-1} + C_2^{-1} \right)^{-1}$. Substituting this and simplifying, we find

$$S = S_{1o} + S_{2o} + C_1 \log \frac{T}{T_{1o}} + C_2 \log \frac{T}{T_{2o}}, \tag{4.1.20}$$

where T is the equilibrium temperature,

$$T = \frac{C_1 T_{1o} + C_2 T_{2o}}{C_1 + C_2}. \tag{4.1.21}$$

If the heat exchange happens through a Carnot engine, then some of the heat extracted from the first block will be converted to work before it is delivered to the second block. If $đQ_1$ is extracted from the first block and $đQ_2$ is delivered to the second block, then reversibility requires

$$\frac{đQ_1}{T_1} = \frac{đQ_2}{T_2} \equiv dA. \tag{4.1.22}$$

Thus, we have

$$dT_1 = -\frac{T_1}{C_1}\, dA, \qquad dT_2 = \frac{T_2}{C_2}\, dA. \tag{4.1.23}$$

Integrating we find $T_1 = T_{1o} e^{-A/C_1}$ and $T_2 = T_{2o} e^{A/C_2}$; setting these two equal, we find an equilibrium temperature

$$T = T_1^{\frac{C_1}{C_1+C_2}} T_2^{\frac{C_2}{C_1+C_2}}. \tag{4.1.24}$$

The total work output of the Carnot engine will be the heat extracted from the warm block minus the heat delivered to the cold block, or

$$W = C_1(T_1 - T) - C_2(T - T_2) = C_1 T_1 + C_2 T_2 - (C_1 + C_2)T. \tag{4.1.25}$$

Problem 4.2 (M98T1)
A Carnot engine uses n moles of an ideal gas as its working substance. The absolute temperatures of its hot and cold reservoirs are denoted by T_1 and T_2, respectively. The net work performed by the engine in one cycle of operation is W. The specific heats of the gas may be assumed independent of the temperature. An investigator is asked to check the values of the reservoir temperatures, but unfortunately she is not provided with a thermometer. However, she is able

to measure W, and also the following volumes:

V_1 = volume of working substance when first contacted with hot reservoir,
V_2 = volume of working substance after extracting heat from hot reservoir,
V_3 = volume of working substance when first contacted with cold reservoir,
V_4 = volume of working substance after giving up heat to cold reservoir.

Derive expressions for the unknown temperatures, T_1 and T_2, in terms of n, W, ratios of the above volumes, the molar gas constant R, and the ratio γ of the constant pressure and constant volume specific heats for the gas.

Thermometers are pretty cheap. Just saying. Anyway, we can express the states of the engine as

$$(p_1, V_1), \left(p_1 \frac{V_1}{V_2}, V_2\right), \left(p_1 \frac{V_1}{V_2}\frac{V_2^\gamma}{V_3^\gamma}, V_3\right), \left(p_1 \frac{V_1}{V_2}\frac{V_2^\gamma}{V_3^\gamma}\frac{V_3}{V_4}, V_4\right). \tag{4.1.26}$$

If make an adiabatic transformation on the final state to close the cycle, we find

$$\left(\frac{V_1 V_3}{V_2 V_4}\right)^{1-\gamma} = 1 \implies V_1 V_3 = V_2 V_4. \tag{4.1.27}$$

The heat extracted from the hot reservoir is

$$Q_H = \int_{V_1}^{V_2} \frac{p_1 V_1}{V}\, dV = p_1 V_1 \log \frac{V_2}{V_1}, \tag{4.1.28}$$

and similarly the heat delivered to the cold reservoir is

$$Q_C = p_1 \frac{V_1}{V_2}\frac{V_2^\gamma}{V_3^\gamma}\frac{V_3}{V_4} V_4 \log \frac{V_3}{V_4}. \tag{4.1.29}$$

Simplifying, we have

$$W = Q_H - Q_C = p_1 V_1 \left(\frac{V_2}{V_3}\right)^{\gamma-1} \log \frac{V_2}{V_1}. \tag{4.1.30}$$

Using the equation of state, we have $p_1 V_1 = nRT_1$, and so we find

$$T_1 = \frac{W}{nR}\left(\frac{V_3}{V_2}\right)^{\gamma-1}\left(\log \frac{V_2}{V_1}\right)^{-1}. \tag{4.1.31}$$

Similarly,

$$T_2 = T_1 \frac{p_3 V_3}{p_1 V_1} = \frac{W}{nR}\left(\log \frac{V_2}{V_1}\right)^{-1}. \tag{4.1.32}$$

We have shown that isolated systems tend to maximize their entropy. We can also express this as

a minimization of the internal energy. Using the identity

$$\left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y = -1, \tag{4.1.33}$$

we have

$$\left(\frac{\partial U}{\partial x}\right)_S = -\left(\frac{\partial S}{\partial x}\right)_U \left(\frac{\partial U}{\partial S}\right)_x = -T\left(\frac{\partial S}{\partial x}\right)_U. \tag{4.1.34}$$

Thus, if entropy is extremized with respect to all degrees of freedom x, so too is the internal energy. Note that this assumes the entropy is held constant, meaning the system is adiabatically isolated.

If the system is coupled in other ways to its environment, then other functions are more appropriate to describe equilibrium. The internal energy satisfies

$$dU = T\,dS - p\,dV, \tag{4.1.35}$$

and so it is well-suited for systems which are adiabatically isolated ($dS = 0$) and which can do no pressure-volume work ($dV = 0$). If instead we have a system coupled to a heat bath at some fixed temperature, it makes more sense to use the Helmholtz free energy $F = U - TS$, which satisfies

$$dF = -S\,dT - p\,dV. \tag{4.1.36}$$

The addition of $-TS$ in order to change $T\,dS$ to $-S\,dT$ is called a Legendre transformation.

We can see the meaning of Helmholtz free energy more directly by thinking about the work a system can perform. If a system is adiabatically isolated, with $đQ = 0$, then the amount of work that can be extracted as a system moves from state A to state B is $U(A) - U(B)$. If instead the system is held at constant temperature, then the available work is

$$-\int_A^B đW = -\int_A^B (dU - T\,dS) = U(A) - U(B) - T(S(A) - S(B)) = F(A) - F(B). \tag{4.1.37}$$

Thus, change in F measures the useful work from a thermally coupled system, just as change in U represents available work from an adiabatically isolated system. If we consider all transformations (not just reversible ones), then this statement becomes an inequality, and imposing $dV = 0$ (so the system does no work) we find $F(B) \leq F(A)$. Thus, F is minimized in equilibrium for a system in a heat bath.

We could also consider an adiabatically isolated system at constant pressure, rather than constant volume. This is accomplished via the Legendre transformation

$$H = U + PV, \tag{4.1.38}$$

defining the enthalphy H, which gives

$$dH = T\,dS + V\,dp. \tag{4.1.39}$$

Finally, we could consider a system in a heat bath at constant pressure. This is described by the Gibbs free energy,

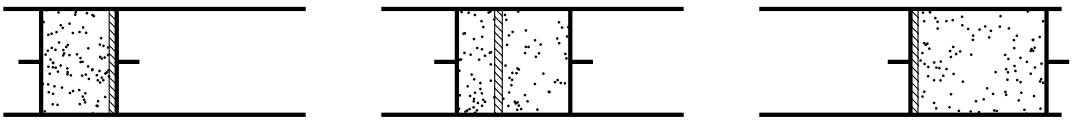
$$G = U + PV - TS, \tag{4.1.40}$$

with

$$dG = -S\,dT + V\,dp. \tag{4.1.41}$$

In their respective situations, H and G are minimized at equilibrium, and differences in these quantities set the maximum amount of work that can be extracted from a system.

Problem 4.3 (M15T2)
The figure below shows a throttling process in which two pistons "push" and "pull" gas through a porous divider (shown as a lightly hatched line) that is fixed inside a thermally insulated cylinder. No heat flows into or out of the cylinder. Initially, all the gas is on the left hand side of the divider and the right hand piston is up against the divider (top left plot). The top right figure shows an intermediate state and the bottom figure shows the final state. The pistons are moved in such a way that the pressure on the left hand side is always P_L and the pressure on the right side is P_R, with $P_R < P_L$. The final volume is larger than the initial volume.



a) What thermodynamic potential has the same value at the end of the process as it did at the start? Prove it!

b) If the gas is ideal, what is its change in the internal energy between initial and final states?

c) Suppose now that the gas has a van der Waals equation of state

$$\left(P + \frac{N^2 a}{V^2}\right)(V - Nb) = NkT \tag{4.1.42}$$

where a and b are small numbers. The Helmholtz free energy is

$$F = -NkT\left\{\ln[n_Q(V - Nb)/N] + 1\right\} - N^2 a/V \tag{4.1.43}$$

where $n_Q = (mkT/2\pi\hbar^2)^{3/2}$. What is the internal energy U?

d) What is the condition on the van der Waals coefficients a and b such that the gas cools on expansion into a very large volume in a throttling process? Qualitatively interpret the result. [Hint: $0 < Nb/V \ll 1$]

No heat flows into or out of the cylinder, so $dS = 0$. The pressure is held constant, so $dp = 0$. Hence,

$$dH = T\,dS + V\,dp = 0, \tag{4.1.44}$$

and enthalpy is constant throughout the process.

Since $H = U + PV$, the change in internal energy is $P_L V_L - P_R V_R$, where V_L and V_R are the initial and final volumes of the left and right chambers, respectively.

We have $U = F + TS$, so in order to compute U we need to compute S. Since

$$dF = -S\,dT - p\,dV, \tag{4.1.45}$$

we have

$$S = -\left(\frac{\partial F}{\partial T}\right)_V = Nk\left(\log\left(n_Q(V - Nb)/N\right) + 1\right) + \frac{3}{2}Nk, \tag{4.1.46}$$

which gives

$$U = \frac{3}{2}NkT - \frac{N^2 a}{V}. \tag{4.1.47}$$

The throttling process holds

$$H = U + pV = \frac{3}{2}NkT + V\left(p - \frac{N^2 a}{V^2}\right) = NkT\left(\frac{3}{2} + \frac{1}{1 - Nb/V}\right) - \frac{2N^2 a}{V} \tag{4.1.48}$$

constant. Since $Nb \ll V$, we can approximate this as

$$H \approx NkT\left(\frac{3}{2} + \frac{N}{V}\left(b - \frac{2a}{kT}\right)\right). \tag{4.1.49}$$

In order to have T decrease as V increases, we must have

$$b < \frac{2a}{kT}. \tag{4.1.50}$$

We can use thermodynamic potentials to derive various identities known as the Maxwell relations. Starting from $dU = T\,dS - p\,dV$, we can use $d^2 = 0$ to write

$$dT \wedge dS = dp \wedge dV. \tag{4.1.51}$$

This equality of volume forms can be expressed in terms of Jacobians:

$$\begin{pmatrix} \left(\frac{\partial T}{\partial a}\right)_b & \left(\frac{\partial T}{\partial b}\right)_a \\ \left(\frac{\partial S}{\partial a}\right)_b & \left(\frac{\partial S}{\partial b}\right)_a \end{pmatrix} = \begin{pmatrix} \left(\frac{\partial p}{\partial a}\right)_b & \left(\frac{\partial p}{\partial b}\right)_a \\ \left(\frac{\partial V}{\partial a}\right)_b & \left(\frac{\partial V}{\partial b}\right)_a \end{pmatrix}, \tag{4.1.52}$$

for any variables a, b. Picking specific examples for a and b, we can find various relations of derivatives. For example, $a = V$ and $b = S$ gives

$$\left(\frac{\partial T}{\partial V}\right)_S = -\left(\frac{\partial p}{\partial S}\right)_V. \tag{4.1.53}$$

Problem 4.4 (M03T1)
An elastic string is found to have the following properties:

- To stretch it to a total length x requires a force $f = \mu x - \alpha T + \beta T x$. Assume that α, β, μ are constants.
- Its heat capacity at constant length x is proportional to temperature: $C_x = A(x)T$.

We can use thermodynamic identities to derive from these facts a variety of other thermal properties. More specifically:

a) Calculate $\left(\frac{\partial S}{\partial x}\right)_T$.

b) Show that A has to be independent of x.

c) Calculate $\left(\frac{\partial S}{\partial T}\right)_x$ and give the general expression for entropy $S(x,T)$ assuming $S(0,0) = B$, where B is a constant.

d) Compute the heat capacity at zero tension $C_F = T\left(\frac{\partial S}{\partial T}\right)_{f=0}$.

For the string, force takes the place of (negative) pressure and length takes the place of volume, so we have

$$dU = T\,dS + f\,dx. \tag{4.1.54}$$

The Maxwell relations then take the form

$$\begin{pmatrix} \left(\frac{\partial T}{\partial a}\right)_b & \left(\frac{\partial T}{\partial b}\right)_a \\ \left(\frac{\partial S}{\partial a}\right)_b & \left(\frac{\partial S}{\partial b}\right)_a \end{pmatrix} = \begin{pmatrix} \left(\frac{\partial x}{\partial a}\right)_b & \left(\frac{\partial x}{\partial b}\right)_a \\ \left(\frac{\partial f}{\partial a}\right)_b & \left(\frac{\partial f}{\partial b}\right)_a \end{pmatrix}. \tag{4.1.55}$$

Picking $a = T$ and $b = x$, this gives

$$\left(\frac{\partial S}{\partial x}\right)_T = -\left(\frac{\partial f}{\partial T}\right)_x = \alpha - \beta x. \tag{4.1.56}$$

We have

$$C_x = \left(\frac{đQ}{dT}\right)_x = T\left(\frac{\partial S}{\partial T}\right)_x. \tag{4.1.57}$$

This means $S = A(x)T + B(x)$. From the previous part, we find

$$A'(x)T + B'(x) = \alpha - \beta x. \tag{4.1.58}$$

Since the right hand side is independent of T, we have $A'(x) = 0$.

We have already shown $\left(\frac{\partial S}{\partial T}\right)_x = A$, and so $S(x,T) = AT + B(x)$. The previous part shows that $B'(x) = \alpha - \beta x$, so we have

$$S(x,T) = AT + B + \alpha x - \frac{1}{2}\beta x^2. \tag{4.1.59}$$

When f is constant, we have

$$\mu\,dx - \alpha\,dT + \beta(T\,dx + x\,dT) = 0 \implies \left(\frac{\partial x}{\partial T}\right)_f = \frac{\alpha - \beta x}{\mu + \beta T}. \tag{4.1.60}$$

Thus,

$$C_F = T\left(A + \frac{(\alpha - \beta x)^2}{\mu + \beta T}\right). \tag{4.1.61}$$

Since $f = 0$ we have $x = \frac{\alpha T}{\mu + \beta T}$, and substituting this gives

$$C_F = AT + \frac{\mu^2\alpha^2 T}{(\mu + \beta T)^3}. \tag{4.1.62}$$

The previous problem shows that much of the formalism is not specific to pressure and volume; any intensive force-like quantity and extensive length-like quantity may enter into the internal energy and other thermodynamic potentials. A common example is the particle number. We define the chemical potential μ as the increase in internal energy when a particle is added to the system, so

$$dU = T\,dS - p\,dV + \mu\,dN. \tag{4.1.63}$$

If we have multiple species in the same system, then they generically have different chemical potentials. If the system is undergoing a phase transition, their chemical potentials must be equal.

Let's assume that V, S, and N are all of the extensive variables on which U depends. If we scale up the system, the energy should scale likewise, so

$$U(\lambda S, \lambda V, \lambda N) = \lambda U(S, V, N). \tag{4.1.64}$$

Taking a derivative with respect to λ, we find

$$U = S\left(\frac{\partial U}{\partial S}\right)_{V,N} + V\left(\frac{\partial U}{\partial V}\right)_{S,N} + N\left(\frac{\partial U}{\partial N}\right)_{V,S} = TS - pV + \mu N. \tag{4.1.65}$$

Taking the differential, we find

$$S\,dT - V\,dp + N\,d\mu = 0. \tag{4.1.66}$$

This is known as the Gibbs-Duhem relation.

Problem 4.5 (J11T3)
We have had a cold December, and it is time for ice skating. The Clausius-Clapeyron equation describes the slope, $\frac{dP}{dT}$, of the 1st-order phase transition line in the pressure-temperature (P, T) phase diagram.

a) Derive the Clausius-Clapeyron Equation for $\frac{dP}{dT}$ in terms of the specific heat and the density difference between the two phases.

b) For the phase change of ice to water, the latent heat of fusion L is about $+3 \times 10^5\,\mathrm{J/kg}$, and the volume change ΔV is about $-10^{-4}\,\mathrm{m^3/kg}$. Estimate the pressure needed to depress the freezing point of ice by 5°C.

c) Comment quantitatively on the urban legend that skates glide with low friction over ice because the ice melts under the pressure of the skate blade pressing down on it. Assume the skater is of mass 70 kg, the skate blade is 30 cm long and 5 mm wide, and the temperature is −5°C.

For the two-phase system, the Gibbs-Duhem relation gives

$$S\,dT - V\,dp + N_1\,d\mu_1 + N_2\,d\mu_2 = 0, \tag{4.1.67}$$

where $N_1 + N_2 = N$ is the total number of particles. At the phase transition we have $\mu_1 = \mu_2$, which means $d\mu_1 = d\mu_2$ along the coexistence curve, which implies

$$\frac{V}{N}\,dp - \frac{S}{N}\,dT = d\mu. \tag{4.1.68}$$

This holds for both phases, so subtracting the relation on each side of the coexistence curve, we find

$$\frac{dp}{dT} = \frac{\Delta S}{\Delta V} = \frac{L}{T\Delta v}, \tag{4.1.69}$$

where v is specific volume.

Substituting the values for water at 273 K, we have

$$\frac{dp}{dT} \approx -1 \times 10^7 \,\mathrm{J/m^3K}, \tag{4.1.70}$$

so we would need about $5 \times 10^7 \,\mathrm{J/m^3}$ of pressure to depress the freezing point by 5°C.

The skater applies a pressure of about $7 \times 10^4 \,\mathrm{J/m^3}$, much less than the requisite pressure we derived from the Clausius-Clapeyron equation to melt the ice. This is strong evidence that a magic ice queen is responsible for ice skating and other ice-related phenomena (see Figure 4.5).

Figure 4.5: The magic ice queen likely to be responsible for the physics of ice skating.

4.2 Canonical Ensemble

Now we seek to derive classical thermodynamics from a microscopic perspective. We start by looking at a system which is completely isolated from its environment. Such a system has a fixed energy U. We describe this system using the microcanonical ensemble, in which the probability of the system being in a state s is given by

$$P(s) = \frac{1}{\Omega} \begin{cases} 1 & \text{if } U(s) = U \\ 0 & \text{otherwise} \end{cases}. \tag{4.2.1}$$

The assumption of a uniform distribution on the space of states with energy U is a fundamental assumption of statistical mechanics. We will find that it works quite well. The factor Ω is inserted for normalization. We define entropy to be

$$S = k_B \log \Omega. \tag{4.2.2}$$

To justify this assumption, suppose we place two such systems in thermal contact. Then the total energy U is fixed, but the subsystems can exchange heat, so they have energies U_1 and $U - U_1$ where U_1 can vary. The total entropy is determined by

$$\Omega = \int_0^U \Omega_1 \times \Omega_2 \, dU_1 = \int_0^U \exp\left(\frac{S(U_1) + S_2(U - U_1)}{k_B}\right) dU_1. \tag{4.2.3}$$

The integrand will be sharply peaked at the equilibrium distribution of energies, where the argument of the exponential is maximized. This gives

$$\frac{\partial S_1}{\partial U_1} = \frac{\partial S_2}{\partial U_2}, \tag{4.2.4}$$

which means $T_1 = T_2$. Thus, we recover the result (technically known as the zeroth law of thermodynamics) that systems are in thermal equilibrium when they have equal temperatures.

Problem 4.6 (M02T3)
One of the all-time classic experiments was the measurement of the residual entropy S of ordinary water-ice by Giauque and Stout. This is the entropy that ice has at a temperature of zero kelvin. This entropy is not zero.

a) What 'law' does this violate? How would you make such an absolute determination of this entropy?

Pauling proposed a model to explain this entropy and the problem here will be to calculate S according to this model. The crystal structure of ice is such that each oxygen atom has 4 nearest neighbor oxygen atoms. On each 'bond' between nearest neighbor oxygen pairs sits a hydrogen atom, but it does not sit in the middle. It sits in one of two positions close to one of the two oxygens at the end of the bond.

b) Assume there are N water molecules. If all hydrogen configurations are equally likely, what is S?

Not all configurations are equally likely. Pauling introduced his "ice rule": There are precisely two hydrogen atoms close to each oxygen atom, reflecting the molecular structure of water. Otherwise, all configurations are equally likely. This limits the number of configurations, but presents a horrific combinatorial problem. Pauling simplified matters further with his Pauling approximation: He estimated the fraction of allowed configurations by taking the hydrogen configuration surrounding one oxygen atom to be independent of those surrounding all other oxygen atoms.

c) Compute S on the basis of the Pauling approximation.

Residual entropy at zero temperature violates the so-called third law of thermodynamics. To measure entropy at zero temperature, one could measure changes in free energy at very low temperatures, since

$$S = -\left(\frac{\partial F}{\partial T}\right)_V. \tag{4.2.5}$$

Since each oxygen atom has four nearest neighbors, but the hydrogen atoms on each are shared between two oxygen atoms, each one individually contributes a factor of $2^2 = 4$ to the number of configurations. Thus, in this first approximation the entropy is

$$S = k_B \log\left(4^N\right) = N k_B \log 4. \tag{4.2.6}$$

Using the Pauling approximation, we find $\binom{4}{2} = 6$ configurations per water molecule, as opposed to the 16 we would have without the Pauling rule, so

$$S = N k_B \log \frac{3}{2}. \tag{4.2.7}$$

We rarely encounter situations where the microcanonical ensemble is appropriate. Much more commonly, we are given the temperature of a system rather than its internal energy. In this case, we can think of the system as being coupled to a large reservoir at temperature T. The sum of the system and the reservoir is isolated, so it can be described in the microcanonical ensemble. Every state of the composite is equally likely, so the probability of a given state is proportional to the corresponding number of states of the reservoir. The first order correction to the entropy of the reservoir due to the energy E of the system is $-\frac{\partial S}{\partial E}E = -\frac{E}{T}$, so the probability of the system having energy E is

$$P(U) = \frac{1}{Z}\exp\left(-\frac{E}{k_B T}\right), \tag{4.2.8}$$

where

$$Z = \sum \exp\left(-\frac{E}{k_B T}\right) \tag{4.2.9}$$

is called the partition function. The sum is over all possible states of the system.

The partition function is surprisingly useful. For example, we can use it to compute the internal energy, which in the canonical ensemble is the expectation value of energy:

$$U = \langle E \rangle = \frac{1}{Z}\sum E \exp\left(-\frac{E}{k_B T}\right) = \frac{k_B T^2}{Z}\frac{\partial Z}{\partial T} = k_B T^2 \frac{\partial(\log Z)}{\partial T}. \tag{4.2.10}$$

It is useful to define $\beta^{-1} = k_B T$, so we can write this as

$$U = -\frac{\partial(\log Z)}{\partial \beta}. \tag{4.2.11}$$

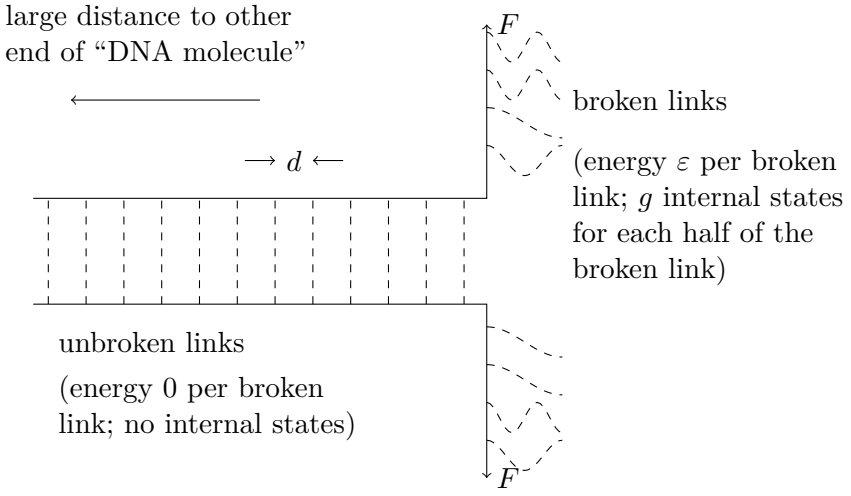
Since $F = U - TS$, we have

$$\frac{\partial(\beta F)}{\partial \beta} = F - T\frac{\partial F}{\partial T} = F + TS = U. \tag{4.2.12}$$

Comparing this with our expression for U in terms of Z, we find

$$F = -k_B T \log Z. \tag{4.2.13}$$

Problem 4.7 (J06T3)
A simple "toy model" model for how complementary strands of DNA are bound together resembles a zipper (see figure). The two strands are connected by "links" (base pairs) spaced at equal intervals d along the strands. It costs an energy ε to break a link, and a link can only be broken if its neighbor to the right is also broken. An unbroken link is a unique internal state, but each of the two dangling ends of a broken link can be one of g internal states.



At the right-hand end of the DNA molecule, the experimenter applies a tension force F to each of the two strands to try to separate them. This force is not strong enough to separate the chains at $T = 0$.

a) Assume that $g = 1$ (so broken links have no internal states). At finite temperatures $k_B T \gg \varepsilon$, what is the mean number $\overline{n}$ of broken links near the end of the DNA molecule, when $F = 0$? (Assume that $\overline{n}d$ is much smaller than the total length L of the DNA molecule.) How does it change when the force is applied?

b) Now assume that $g > 1$. Write down the configurational partition function, and obtain the free energy associated with the links between the strands. Obtain the critical temperature $T_c(g, \varepsilon, F, D)$ above which the two strands of an infinitely long DNA molecule would be pulled apart by the applied force F.

c) Obtain an expression for $\overline{n}(T, g, \varepsilon, F, d)$ valid for an infinitely-long DNA molecule at all temperatures less than T_c (including $k_B T \ll \varepsilon$), and make a sketch showing its principal features.

The partition function is given by

$$Z = \sum_n e^{-n\varepsilon/k_B T} = \frac{1}{1 - e^{-\varepsilon/k_B T}}. \tag{4.2.14}$$

The mean number of broken links is

$$\overline{n} = \frac{1}{Z}\sum_n n e^{-n\varepsilon/k_BT} = -k_BT\frac{\partial(\log Z)}{\partial\varepsilon} = \frac{e^{-\varepsilon/k_BT}}{1-e^{-\varepsilon/k_BT}}. \tag{4.2.15}$$

When force is applied, the energy cost of breaking a link changes to $\varepsilon - Fd$, and so the partition function is

$$Z = \frac{1}{1-e^{(Fd-\varepsilon)/k_BT}}. \tag{4.2.16}$$

For $g > 1$, the partition function becomes

$$Z = \sum_n g^{2n} e^{n(Fd-\varepsilon)/k_BT} = \frac{1}{1-g^2e^{(Fd-\varepsilon)/k_BT}}. \tag{4.2.17}$$

The free energy is then

$$F = k_BT\log\left(1-g^2e^{(Fd-\varepsilon)/k_BT}\right). \tag{4.2.18}$$

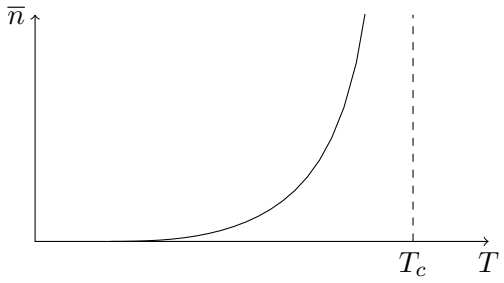
The critical temperature is then

$$T_c = \frac{\epsilon - Fd}{2k_B\log g}. \tag{4.2.19}$$

The expected number of broken links is

$$\overline{n} = \frac{\partial F}{\partial\varepsilon} = \frac{g^2e^{(Fd-\varepsilon)/k_BT}}{1-g^2e^{(Fd-\varepsilon)/k_BT}}. \tag{4.2.20}$$

The behavior as a function of T is shown below, for $\epsilon > Fd$. When $\epsilon < Fd$, the molecule is only stable at high temperatures.



Problem 4.8 (J14T3)
Organic polymers are modeled as flexible chains whose links are rigid segments of length b that can pivot freely relative to each other. In the random walk approximation, the effects of overlaps between the links are ignored and the polymer configurations are taken to resemble random paths of N steps.

Take it as given that, for a simple random walk:

i) the end to end distance $R(N)$ scales as $R(N) \approx b\sqrt{N}$.

ii) the probability of landing at $\boldsymbol{R}$ after N steps (starting at the origin) is $\approx e^{-R^2/(2Nb^2)}$ (up to irrelevant pre-factors).

a) At the level of the random walk approximation, what is the entropy of the idealized polymer of N units with total length R?

b) The polymer's self energy is modeled by a (repulsive) energy $\lambda > 0$ for any two units that come within a fixed distance from each other (and zero contribution otherwise). Assuming that the polymer's units are spread relatively uniformly over a volume of diameter R, obtain an approximate expression for the polymer's free energy showing the dependence on λ, b, N, R and T (powers or simple functions). (You do not need to specify the constant coefficients.)

c) Minimizing the free energy, derive a relation of the form: $\langle R\rangle \approx N^{\nu}$ for the order of magnitude estimate of the equilibrium end-to-end distance of the self-repelling polymer (at fixed b and $\lambda > 0$), in d dimensions. What value does the approximate expression for the free energy yield for the exponent ν?

Each unit can be oriented in any direction, so there are $(4\pi)^N$ total ways to orient N units. We are given that the probability of the walk ending at $\boldsymbol{R}$ is proportional to $e^{-R^2/(2Nb^2)}$, so the probability of having total length R scales as $R^2 e^{-R^2/(2Nb^2)}$. Thus, we have

$$\Omega = (4\pi)^N \frac{R^2 e^{-R^2/2Nb^2}}{\int_0^\infty x^2 e^{-x^2/2Nb^2}\,dx} = 2^{2N+1/2}\pi^{N+1/2} R^2 b^{-3} N^{-3/2} e^{-R^2/2Nb^2}. \tag{4.2.21}$$

The entropy is

$$S = k_B \log\Omega \approx k_B\left(N\log(4\pi) + \frac{1}{2}\log(2\pi) + 2\log R - 3\log b - \frac{3}{2}\log N - \frac{R^2}{2Nb^2}\right). \tag{4.2.22}$$

Since the rest of the problem is mostly qualitative, we may focus on the dominant term here,

$$S \approx -k_B \frac{R^2}{2Nb^2}. \tag{4.2.23}$$

This term would be the same in any number of dimensions d.

The density of units scales as NR^{-d} so the self energy scales as

$$U = \alpha\lambda N^2 R^{-d}. \tag{4.2.24}$$

Thus, the free energy is

$$F = U - TS = \alpha N^2 R^{-d} - k_B T \frac{R^2}{2Nb^2}. \tag{4.2.25}$$

Minimizing the free energy with respect to R gives

$$-d\alpha N R^{-d-1} - \frac{2k_B T R}{2Nb^2} = 0 \implies \langle R\rangle \sim N^{\frac{3}{d+2}}, \tag{4.2.26}$$

so $\nu = \frac{3}{d+2}$. Note that $\nu = 1$ in one dimension, as we should expect for a self-repelling polymer on a line.

Problem 4.9 (J09T2)
In a lattice of N sites, each site is occupied by an atom at zero temperature. A lattice defect occurs when an atom moves to an interstitial site. The energy cost of a defect is Δ. At finite temperature T, we expect a finite number $\langle n(T)\rangle$ of defects to exist in equilibrium. Assume that defects do not interact with each other.

a) Write down an expression for the partition function Z.

b) Calculate $\langle n\rangle$ and the total free energy F of the lattice from Z at temperature T.

c) Find the entropy $S(T)$ and heat capacity C_V from F.

d) Use a purely statistical argument to rederive the entropy S starting with the total number of configurations W_n with n defects. Using your answer for $\langle n\rangle$, show that S agrees with part c).

e) Use physical arguments to reproduce your answer for C_V in the low T limit ($\beta\Delta \gg 1$, where $\beta = 1/k_BT$).

The partition function is given by

$$Z = \sum_{n=0}^{N} \binom{N}{n} e^{n\Delta/k_BT} = \left(1 + e^{\Delta/k_BT}\right)^N. \tag{4.2.27}$$

The expected number of defects is

$$\langle n\rangle = -\frac{1}{\Delta}\frac{\partial(\log Z)}{\partial\beta} = \frac{N}{1 + e^{\Delta/k_BT}}. \tag{4.2.28}$$

The free energy is

$$F = -k_BT \log Z = -Nk_BT \log\left(1 + e^{\Delta/k_BT}\right). \tag{4.2.29}$$

The entropy is

$$S = -\frac{\partial F}{\partial T} = Nk_B \log\left(1 + e^{\Delta/k_BT}\right) - \frac{\Delta N/T}{1 + e^{-\Delta/k_BT}}. \tag{4.2.30}$$

The heat capacity is

$$C_V = T\frac{dS}{dT} = \frac{N\Delta^2}{4k_BT^2}\operatorname{sech}^2\left(\frac{\Delta}{2k_BT}\right). \tag{4.2.31}$$

The total number of configurations with n defects is $W_n = \binom{N}{n}$, so the entropy of a state with n defects is

$$S = k_B \log\binom{N}{n} \approx Nk_b \log N - nk_B \log n - (N-n)k_B \log(N-n). \tag{4.2.32}$$

in the large N limit. Substituting (4.2.28), we find

$$\begin{aligned} S &= \frac{Nk_B}{1+e^{\Delta/k_BT}} \log\left(1+e^{\Delta/k_BT}\right) + \frac{Nk_Be^{\Delta/k_BT}}{1+e^{\Delta/k_BT}} \log\left(1+e^{-\Delta/k_BT}\right) \\ &= Nk_B \log\left(1+e^{\Delta/k_BT}\right) - \frac{\Delta N/T}{1+e^{-\Delta/k_BT}}, \end{aligned} \tag{4.2.33}$$

which agrees with the result obtained from the partition function.

At very low temperatures, we have $n = Ne^{-\beta\Delta}$, and so the energy will be $U = N\Delta e^{-\beta\Delta}$. Then the specific heat will be $\frac{dU}{dT} = Nk_B(\beta\Delta)^2e^{-\beta\Delta}$. Indeed, in the low temperature limit of (4.2.31), we have

$$C_V = Nk_B(\beta\Delta)^2e^{-\beta\Delta}. \tag{4.2.34}$$

What about gases? In principle we should be able to derive everything about the ideal gas by modeling it as a collection of N identical noninteracting particles in a volume V. We can get away with working in the microcanonical ensemble in this case, so we fix the total energy at U. Then the momenta of the particles satisfy $\sum p^2 = 2mU$, which specifies a sphere of radius $\sqrt{2mU}$ embedded in $3N$-dimensional space. Just in case you ever find yourself with a gun to your head and asked to write down the volume of hyperspheres in N dimensions, you can do it by integrating Gaussians in both Cartesian and spherical coordinates:

$$\int_{\mathbb{R}^d} e^{-r^2}\, d^d\boldsymbol{x} = \pi^{d/2}, \tag{4.2.35}$$

$$\int_{\mathbb{R}^d} e^{-r^2}\, d^d\boldsymbol{x} = \mathrm{vol}\left(S^{d-1}\right) \int_0^\infty r^{d-1}e^{-r^2}\, dr = \frac{\Gamma(d/2)}{2}\, \mathrm{vol}\left(S^{d-1}\right). \tag{4.2.36}$$

This shows that $\mathrm{vol}(S^{d-1}) = \frac{2\pi^{d/2}}{\Gamma(d/2)}$. This means the volume available in momentum phase space is $(2mU)^{(3N-1)/d}\frac{2\pi^{3N/2}}{\Gamma(3N/2)}$. Multiplying by the volume factor, we have

$$S = k_B \log\Omega = k_B \log\left(V^N(2mU)^{(3N-1)/2}\frac{2\pi^{3N/2}}{\Gamma(3N/2)}\right). \tag{4.2.37}$$

In the large N limit, this becomes

$$S = Nk_B \log\left(V\left(\frac{4\pi mU}{3N}\right)^{3/2}\right). \tag{4.2.38}$$

We can now extract some basic results of thermodynamics for an ideal gas. We have

$$\frac{1}{T} = \frac{\partial S}{\partial U} = \frac{3Nk_B}{2U}, \tag{4.2.39}$$

so the energy is $\frac{3}{2}Nk_BT$. To derive the equation of state, we evaluate

$$\frac{p}{T} = \frac{\partial S}{\partial V} = \frac{Nk_B}{V}, \tag{4.2.40}$$

so indeed $pV = Nk_BT$. We can also learn some things that aren't so obvious from thermodynamics. Perhaps most famously, we can work out the momentum distribution for a single particle. If we fix

the momentum of one particle at $\boldsymbol{p}$, then the other $3N-1$ components should have energy $U - \frac{p^2}{2m}$. The residual entropy is

$$S' = k_B \log\left(V^N (2mU - p^2)^{(3N-4)/2} \frac{2\pi^{(3N-4)/2}}{\Gamma(3(N-1)/2)} \right), \tag{4.2.41}$$

and the probability is

$$P(\boldsymbol{p}) = e^{(S'-S)/k_B} \tag{4.2.42}$$

$$= \frac{1}{(2\pi mU)^{3/2}} \left(\frac{2mU - p^2}{2mU} \right)^{(3N-4)/2} \frac{\Gamma(3N/2)}{\Gamma(3(N-1)/2)} \tag{4.2.43}$$

$$= \frac{1}{(3\pi mUN)^{3/2}} \left(1 - \frac{p^2}{2mU} \right)^{(3N-4)/2}, \tag{4.2.44}$$

where we have used the large N limit to compute the ratio of the Γ functions. Now, using $U = \frac{3}{2} N k_B T$ and once again invoking large N, we find

$$P(\boldsymbol{p}) = \left(\frac{1}{2\pi m k_B T} \right)^{3/2} \exp\left(-\frac{p^2}{2mk_B T} \right). \tag{4.2.45}$$

This is the Maxwell-Boltzmann distribution, and using it we can work out just about anything about the kinematics of ideal gas particles.

Problem 4.10 (J15T3)
Consider a classical ideal gas of identical non-interacting particles of mass m in a container of volume V at initial temperature T_i. Let the particles have spin one-half and magnetic moment μ, and let the container be placed in a strong magnetic field H.

a) Compute the classical partition function for this system, taking proper account of particle identity. It may help you to know that the partition function for a single classical particle (ignoring the spin degree of freedom) is $Z_1 = n_Q V$ where $n_Q = (mkT/2\pi\hbar^2)^{3/2}$.

b) Calculate the total energy and entropy for this system.

c) Now suppose that the container is thermally isolated and that the magnetic field is slowly reduced (i.e. adiabatically). Show that the temperature decreases continuously as H is decreased.

d) Show that if H is reduced all the way to zero the final temperature satisfies the inequality $T_i > T_f > 2^{-2/3} T_i$.

From quantum mechanics, we know that the two spin eigenstates will have energies $\pm\frac{\mu H}{2}$; since we are working classically, we will treat these eigenstates as the only states. The partition function is then

$$Z = \frac{1}{N!} \left(2 n_Q V \cosh\left(\frac{\mu H}{2k_B T} \right) \right)^N, \tag{4.2.46}$$

where the factor of $\frac{1}{N!}$ accounts for the particles being identical.

The energy is

$$U = -\frac{\partial(\log Z)}{\partial\beta} = \frac{3}{2}Nk_BT - \frac{N\mu H}{2}\tanh\left(\frac{\mu H}{2k_BT}\right). \tag{4.2.47}$$

The entropy is

$$\begin{aligned} S &= -\frac{\partial F}{\partial T} = k_B \log Z + k_B T \frac{\partial(\log Z)}{\partial T} \\ &= Nk_B \log\left(\frac{2n_Q V}{N}\cosh\left(\frac{\mu H}{2k_BT}\right)\right) + \frac{3}{2}Nk_B - \frac{N\mu H}{2T}\tanh\left(\frac{\mu H}{2k_BT}\right). \end{aligned} \tag{4.2.48}$$

To understand adiabatic demagnetization, we need to compute $\left(\frac{\partial T}{\partial H}\right)_S$. Expressing this as

$$\left(\frac{\partial T}{\partial H}\right)_S = -\frac{\left(\frac{\partial S}{\partial H}\right)_T}{\left(\frac{\partial S}{\partial T}\right)_H}, \tag{4.2.49}$$

we can grind through derivatives and find

$$\left(\frac{\partial T}{\partial H}\right)_S = \frac{\mu^2 T H}{3k_B^2T^2 + \mu^2H^2 + 3k_B^2T^2\cosh\left(\frac{\mu H}{k_BT}\right)}. \tag{4.2.50}$$

That's fugly. Put it in the burn book. But it is continuous as $H \to 0$, so the temperature decreases continuously.

Setting the entropies at (H, T_i) and $(0, T_f)$ equal, we find

$$\frac{T_f}{T_i} = \exp\left(-\frac{2}{3}x\tanh x\right)\cosh^{2/3}(x) = \left(\frac{e^{-x(1+\tanh x)} + e^{x(1-\tanh x)}}{2}\right)^{2/3}, \tag{4.2.51}$$

where $x = \frac{\mu H}{2k_BT_i}$. At $x = 0$ we have $T_f = T_i$, and as $x \to \infty$, $\tanh x \to 1$ and so we have $T_f \to 2^{-2/3}T_i$. It should be relatively straightforward to show that $\frac{T_f}{T_i}$ decreases monotonically, so 1 and $2^{-2/3}$ are in fact its upper and lower bounds for $x \geq 0$.

4.3 Applications to Classical Physics

As it turns out, a whole lot of systems contain more than 65,536 particles, so statistical physics comes up a lot. Some of these applications involve tools other than the canonical ensemble, but not substantially different.

We begin with one of these slightly different tools, the grand canonical ensemble. Contrary to popular belief, this is not a group of string instrumentalists who play for the Pope, but rather a generalization of the canonical ensemble for systems with a variable number of particles. Much like the canonical ensemble, we think of our system as being coupled to a large reservoir, but now the system and reservoir can exchange both heat and particles. The probability for a microstate of the

system is determined by the decrease in entropy of the reservoir:

$$p \propto \exp\left(dS/k_B\right) = \exp\left(\frac{\mu N - E}{k_B T}\right). \tag{4.3.1}$$

The normalization factor is the grand partition function,

$$\mathcal{Z} = \sum \exp\left(\frac{\mu N - E}{k_B T}\right). \tag{4.3.2}$$

We can use this to obtain expectation values; for example,

$$\langle N \rangle = \frac{1}{\beta}\frac{\partial(\log \mathcal{Z})}{\partial \mu}. \tag{4.3.3}$$

We can compute the grand partition function in terms of the regular partition function by rewriting the sum:

$$\mathcal{Z} = \sum_{N=0}^{\infty} e^{\beta \mu N} Z(N,T). \tag{4.3.4}$$

In the thermodynamic limit, fluctuations around $\langle N \rangle$ are relatively small, so we can focus on the term $N = \langle N \rangle$ in the sum, and find

$$-k_B T \log \mathcal{Z} = F - \mu N = U - TS - \mu N. \tag{4.3.5}$$

We define the grand potential

$$\mathcal{G} = U - TS - \mu N, \tag{4.3.6}$$

so that

$$\mathcal{G} = -k_B T \log \mathcal{Z} \tag{4.3.7}$$

in analogy to the relation between Z and F. All these fancy letters, so grand.

Problem 4.11 (J99T1)
Suppose a new kind of particle is discovered. This particle is known as the weirdon since it obeys weird statistics in which a given state may contain 0, 1, or 2 particles. Furthermore, weirdons are one dimensional and we will be considering a gas of non-interacting weirdons confined to a straight line of length L. The weirdons are weakly coupled to a thermal reservoir at temperature τ and the weirdon mass is m.

a) Suppose the chemical potential of the weirdons is μ. What is the occupancy of a state with energy ϵ? In addition, give numerical values of the occupancy for $(\mu - \epsilon)/\tau = -\infty$, $(\mu - \epsilon)/\tau = 0$, and $(\mu - \epsilon)/\tau = +\infty$.

b) What is the density of states? (That is, the number of states per unit energy as a function of energy?) Remember, the weirdons are one dimensional and are confined to a "box" of length L.

c) Suppose the weirdon gas is cold ($\tau \to 0$) and contains N weirdons. What is the chemical potential?

d) Under the same conditions as part c), what is the total energy of the weirdon gas? Be sure to eliminate μ from your expression.

e) The low temperature heat capacity of the weirdon gas is proportional to the temperature to some power, $C \propto \tau^\alpha$. What is α?

Let's imagine the state with energy ϵ is the sole state. Then we have three microstates, and the grand partition function is

$$\mathcal{Z} = 1 + e^{\beta(\mu-\epsilon)} + e^{2\beta(\mu-\epsilon)}. \tag{4.3.8}$$

The occupancy is

$$\langle N \rangle = \frac{1}{\beta}\frac{\partial(\log \mathcal{Z})}{\partial \mu} = \frac{e^{\beta(\mu-\epsilon)} + 2e^{2\beta(\mu-\epsilon)}}{1 + e^{\beta(\mu-\epsilon)} + e^{2\beta(\mu-\epsilon)}}. \tag{4.3.9}$$

For $\beta(\mu-\epsilon) = -\infty$, we have $\langle N \rangle = 0$. For $\beta(\mu-\epsilon) = 0$, we have $\langle N \rangle = 1$. For $\beta(\mu-\epsilon) = +\infty$, we have $\langle N \rangle = 2$.

Since the weirdons are particles in a box, their energy levels are

$$\epsilon_n = \frac{\pi^2\hbar^2 n^2}{2mL^2}. \tag{4.3.10}$$

The density of states is

$$g(\epsilon) = \frac{dn}{d\epsilon} = \frac{L}{2\pi\hbar}\sqrt{\frac{2m}{\epsilon}}. \tag{4.3.11}$$

The number of weirdons can be approximated as

$$N = \int_0^\infty d\epsilon\, g(\epsilon)\frac{e^{\beta(\mu-\epsilon)} + 2e^{2\beta(\mu-\epsilon)}}{1 + e^{\beta(\mu-\epsilon)} + e^{2\beta(\mu-\epsilon)}}. \tag{4.3.12}$$

As $\beta \to \infty$, the occupation number will be 0 for $\epsilon > \mu$ and 2 for $\epsilon < \mu$. Thus, we have

$$N = 2\int_0^\mu d\epsilon\, g(\epsilon) = \frac{2L}{\pi\hbar}\sqrt{2m\mu}. \tag{4.3.13}$$

Solving for the chemical potential, we have

$$\mu = \frac{N^2\pi^2\hbar^2}{8mL^2}. \tag{4.3.14}$$

The total energy is given by

$$U = 2\int_0^\mu d\epsilon\, \epsilon g(\epsilon) = \frac{2L}{3\pi\hbar}\sqrt{2m\mu^3} = \frac{N\mu}{3} = \frac{N^3\pi^2\hbar^2}{24mL^2}. \tag{4.3.15}$$

To find the low temperature heat capacity, we write the energy in all its glory,

$$U = \int_0^\infty d\epsilon\, \epsilon g(\epsilon)\frac{e^{\beta(\mu-\epsilon)} + 2e^{2\beta(\mu-\epsilon)}}{1 + e^{\beta(\mu-\epsilon)} + e^{2\beta(\mu-\epsilon)}}. \tag{4.3.16}$$

One could take a derivative, in the same way that one could slap oneself across the face. Whatever floats your boat. However, it will be quicker to develop an expansion in β^{-1} and look for the lowest non-constant term. We already know the constant term, so the rest is

$$U - U(\beta = \infty) = \int_0^\mu d\epsilon\, \epsilon g(\epsilon) \left(\frac{e^{\beta(\mu-\epsilon)} + 2e^{2\beta(\mu-\epsilon)}}{1 + e^{\beta(\mu-\epsilon)} + e^{2\beta(\mu-\epsilon)}} - 2 \right) + \int_\mu^\infty d\epsilon\, \epsilon g(\epsilon) \frac{e^{\beta(\mu-\epsilon)} + 2e^{2\beta(\mu-\epsilon)}}{1 + e^{\beta(\mu-\epsilon)} + e^{2\beta(\mu-\epsilon)}}. \tag{4.3.17}$$

Some convenient simplification of the first term gives

$$U - U(\beta = \infty) = -\int_0^\mu d\epsilon\, \epsilon g(\epsilon) \frac{e^{-\beta(\mu-\epsilon)} + 2e^{-2\beta(\mu-\epsilon)}}{1 + e^{-\beta(\mu-\epsilon)} + e^{-2\beta(\mu-\epsilon)}} + \int_\mu^\infty d\epsilon\, \epsilon g(\epsilon) \frac{e^{\beta(\mu-\epsilon)} + 2e^{2\beta(\mu-\epsilon)}}{1 + e^{\beta(\mu-\epsilon)} + e^{2\beta(\mu-\epsilon)}}. \tag{4.3.18}$$

These integrals look very similar, and we can make them moreso. We extend the domain of the first integral to $-\infty$ by setting $g(\epsilon < 0) = 0$, and then make the change of variables $x = \pm\beta(\mu - \epsilon)$ (different signs in the two integrals), so

$$\begin{aligned} U - U(\beta = \infty) = &-\beta^{-1} \int_{-\infty}^0 dx\, \left(\mu + \beta^{-1}x\right) g\left(\mu + \beta^{-1}x\right) \frac{e^x + 2e^{2x}}{1 + e^x + e^{2x}} \\ &+ \beta^{-1} \int_{-\infty}^0 dx\, \left(\mu - \beta^{-1}x\right) g\left(\mu - \beta^{-1}x\right) \frac{e^x + 2e^{2x}}{1 + e^x + e^{2x}}. \end{aligned} \tag{4.3.19}$$

We can then add the integrals. Both integrals are sharply peaked near $x = 0$, so we can write the difference as a derivative and find

$$U = U(\beta = \infty) + \beta^{-2} \left(\frac{d}{d\epsilon}(\epsilon g(\epsilon)) \right)_{\epsilon=\mu} \int_{-\infty}^0 dx\, \frac{e^x + 2e^{2x}}{1 + e^x + e^{2x}}. \tag{4.3.20}$$

We could stop here: the integral is just a number, so the leading order correction to the energy is order T^2, meaning $C \propto T$ for small T. But we might as well finish off, since the integral is pretty trivial to evaluate:

$$\int_{-\infty}^0 dx\, \frac{e^x + 2e^{2x}}{1 + e^x + e^{2x}} = \int_{-\infty}^0 \frac{d}{dx} \left(\log\left(1 + e^x + e^{2x}\right) \right) dx = \log 3. \tag{4.3.21}$$

Thus, we have

$$C = \left(\frac{2k_B^2 \log 3}{\pi^2 \hbar^2} \frac{mL^2}{N} \right) T + \mathcal{O}(T^2). \tag{4.3.22}$$

Problem 4.12 (J01T2)
Imagine a solution of three types of biomolecules. Type A and type C molecules can form a bound system with energy $-\epsilon_{AC}$ ($\epsilon_{AC} > 0$) relative to $\epsilon = 0$ when they are unbound. Similarly, type B and type C molecules bind with energy $-\epsilon_{BC}$. Only one A or B molecule can bind to a C molecule at a time. Further, ϵ_{AC} and ϵ_{BC} are substantially larger than the energies with which other molecules might be bound at the same place on the type C molecule. The solution is an infinite reservoir of A and B molecules as far as the C molecules are concerned.

a) Determine the grand partition function for this system. Also determine the fractions f_A and

f_B of C molecules which have bound an A or a B molecule. You may introduce the chemical potentials, μ_A and μ_B, of A and B molecules.

b) The concentration n_A of A molecules is sufficiently high that in the absence of B molecules, essentially every C molecule binds an A molecule. Obtain an expression for f_A that depends only on the concentrations n_A, n_B, the energies ϵ_{AC} and ϵ_{BC}, and the temperature T the solution.

c) As already remarked, in the absence of B molecules, f_A is close to 1. However, when the concentration n_B of B molecules in solution reaches 1% that of the A molecules in solution, it is observed that f_A drops to 0.1. What is the numerical value of $(\epsilon_{BC} - \epsilon_{AC})/kT$? Make a rough estimate of $\epsilon_{BC} - \epsilon_{AC}$ in electron volts if $T = 300\,\mathrm{K}$.

The grand partition function is

$$\begin{aligned}\mathcal{Z} &= \sum_{n_A,n_B=0}^{N} \frac{N!}{n_A!\,n_B!\,(N-n_A-n_B)!} \exp\left(\beta\left(n_A(\mu_A+\epsilon_{AC}) + n_B(\mu_B+\epsilon_{BC})\right)\right) \\ &= \left(1 + e^{\beta(\mu_A+\epsilon_{AC})} + e^{\beta(\mu_B+\epsilon_{BC})}\right)^N,\end{aligned} \tag{4.3.23}$$

where N is the number of C molecules. The bound fractions are

$$f_A = \frac{\langle n_A\rangle}{N} = \frac{1}{N\beta}\frac{\partial(\log\mathcal{Z})}{\partial\mu_A} = \frac{e^{\beta(\mu_A+\epsilon_{AC})}}{1 + e^{\beta(\mu_A+\epsilon_{AC})} + e^{\beta(\mu_B+\epsilon_{BC})}}, \tag{4.3.24}$$

$$f_B = \frac{\langle n_B\rangle}{N} = \frac{1}{N\beta}\frac{\partial(\log\mathcal{Z})}{\partial\mu_B} = \frac{e^{\beta(\mu_B+\epsilon_{BC})}}{1 + e^{\beta(\mu_A+\epsilon_{AC})} + e^{\beta(\mu_B+\epsilon_{BC})}}. \tag{4.3.25}$$

We are given that when B molecules are absent ($\mu_B \to \infty$), $f_A \to 1$. This means $e^{\beta(\mu_A+\epsilon_{AC})} \gg 1$. We can thus ignore the 1 in the denominator, and so we have

$$f_A = \frac{1}{1 + e^{\beta(\mu_B-\mu_A+\epsilon_{BC}-\epsilon_{AC})}}. \tag{4.3.26}$$

In the classical limit, the concentrations are $n_A \propto e^{\beta\mu_A}$ and $n_B \propto e^{\beta\mu_B}$, so

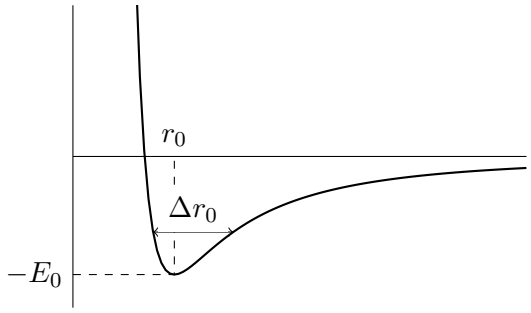
$$f_A = \frac{1}{1 + \frac{n_B}{n_A}e^{\beta(\epsilon_{BC}-\epsilon_{AC})}}. \tag{4.3.27}$$

The given condition is

$$0.1 = \frac{1}{1 + 0.01e^{\beta(\epsilon_{AC}-\epsilon_{BC})}} \implies e^{\beta(\epsilon_{BC}-\epsilon_{AC})} = 900. \tag{4.3.28}$$

This gives $\epsilon_{BC} - \epsilon_{AC} = 0.18\,\mathrm{eV}$ at $300\,\mathrm{K}$.

Problem 4.13 (J03T2)
Krypton atoms are rather heavy and reasonably polarizable. The potential between two krypton atoms is shown in the figure



In the limit in which the mass of the Kr atom is very large, there will at low energies and temperatures be an equilibrium of the form

$$\mathrm{Kr} + \mathrm{Kr} \leftrightarrow \mathrm{Kr}_2 \tag{4.3.29}$$

a) The classical partition function of two krypton atoms inside a volume V can be written as

$$Z_2 = \left(1 + \frac{K}{V}\right) Z_2^{\mathrm{id}} \tag{4.3.30}$$

where Z_2^{id} is the partition sum of two free atoms. How is the constant K related to the probability that the two atoms form a molecule? Find an approximate expression for K in terms of the reaction energy E_0, the size r_0 of the molecule, and the width Δr_0 of the potential.

b) Show that the partition function Z_N for N krypton atoms inside a volume V can similarly be written as a sum of contributions coming from M Kr_2 molecules and $N - 2m$ unbound free Kr atoms given by

$$Z_{N,M} = d(M,N) \left(\frac{K}{V}\right)^M Z_N^{\mathrm{id}} \tag{4.3.31}$$

where Z_N^{id} is the ideal gas partition sum, $d(M,N)$ is the number of ways M molecules can be formed out of N atoms, and K is the same quantity found in part a). Determine the combinatorial factor $d(M,N)$.

c) Derive the equilibrium condition

$$c_{\mathrm{Kr}_2} = K[c_{\mathrm{Kr}}]^2 \tag{4.3.32}$$

where c_{Kr_2} is the concentration of the Kr_2 molecules and c_{Kr} the concentration of the unbound Kr atoms. You may use Stirling's formula $N! \sim \left(\frac{N}{e}\right)^N$.

The classical partition function will be the sum of Z_2^{id} and the piece corresponding to molecule states. The number of states of two krypton atoms is proportional to V^2, while the number of states of the molecule is proportional to V multiplied by the volume which can be occupied by the second krypton atom relative to the first; this is approximately $4\pi r_0^2 \Delta r$. Thus,

$$K = 4\pi r_0^2 \Delta r e^{\beta E_0}. \tag{4.3.33}$$

The probability of the two atoms forming a molecule is

$$P = \frac{\frac{K}{V} Z_2^{\text{id}}}{\left(1 + \frac{K}{V}\right) Z_2^{\text{id}}} = \frac{K}{K+V}. \tag{4.3.34}$$

Every time we bind two atoms into a molecule, the phase space available is reduced by a factor $\frac{K}{V}$ and the energy decreases by E_0. This justifies the given form of $Z_{N,M}$. By elementary combinatorics, we have

$$d(M,N) = \binom{N}{2M} \frac{(2M)!}{2^M \cdot M!} = \frac{N!}{2^M \cdot M!(N-2M)!}. \tag{4.3.35}$$

The total partition function is

$$Z_N = Z_N^{\text{id}} \sum_{M=0}^{N/2} \frac{N!}{2^M (N-2M)!} \left(\frac{K}{V}\right)^M. \tag{4.3.36}$$

This gives a free energy

$$F_N = F_N^{\text{id}} - k_B T \log \left(\sum_{M=0}^{N/2} \frac{N!}{2^M \cdot M!(N-2M)!} \left(\frac{K}{V}\right)^M \right). \tag{4.3.37}$$

In the thermodynamic limit, the sum will be dominated by the term corresponding to the expected number M of molecules which minimizes free energy. Thus,

$$0 = \frac{d}{dM} \log \left(\frac{N!}{2^M \cdot M!(N-2M)!} \left(\frac{K}{V}\right)^M \right) \approx \log \frac{K}{2V} + 2\log(N-2M) - \log M. \tag{4.3.38}$$

This gives $\frac{(N-2M)^2}{M} = \frac{2V}{K}$, or

$$c_{\text{Kr}_2} = \frac{M}{V} = \frac{K}{2} \frac{(N-2M)^2}{V^2} = \frac{K}{2} [c_{\text{Kr}}]^2. \tag{4.3.39}$$

Hmm…close. I think this is right, actually. Also, $2 = 1$, so it doesn't matter.

Now for something entirely unrelated: the equipartition theorem. We showed in the previous section that the internal energy of a monatomic ideal gas is $U = \frac{3}{2} N k_B T$. The factor of $\frac{3}{2}$ comes from the assumption of monatomicity, and if the molecules in the gas have additional degrees of freedom beyond translation, the overall factor will change. The equipartition theorem will give a way to understand this without needing to compute the entropy directly.

Let's assume the energy of the system has a piece which depends quadratically on some continuous variable x_1; i.e.,

$$E = ax_1^2 + f(x_2, x_3, \ldots). \tag{4.3.40}$$

Furthermore, let's assume the system is connected to a heat bath at temperature T, so we can use the canonical ensemble. To find the expectation of the energy in the x_1 degree of freedom, we use the partition function to write

$$\left\langle ax_1^2 \right\rangle = -\frac{\partial}{\partial\beta}\left(\log \sum e^{-\beta E}\right) = -\frac{\partial}{\partial\beta}\left(\log \int dx_1\, e^{-\beta a x_1^2}\right), \tag{4.3.41}$$

where we have used the form of the energy to factor out the piece not depending on x_1. The integral is easily evaluated, and we find

$$\left\langle ax_1^2 \right\rangle = \frac{1}{2\beta} = \frac{k_B T}{2}. \tag{4.3.42}$$

This is the equipartition theorem: for a system in equilibrium with a heat bath at temperature T, every quadratic degree of freedom contributes $\frac{1}{2}k_B T$ to the energy. This immediately reproduces the internal energy of the monatomic ideal gas. Furthermore, if we have a diatomic gas, there are two additional rotational degrees of freedom, so $U = \frac{5}{2}Nk_B T$.

Problem 4.14 (J14T1)
Consider a liquid placed in a very wide container that is in thermal equilibrium at temperature T with its surroundings. Let $z(\boldsymbol{r})$ be the height of the liquid at point $\boldsymbol{r} = (x, y)$ defined such that the equilibrium height in absence of thermal fluctuations is $z(\boldsymbol{r}) = 0$. For small deviations around the equilibrium, the potential energy is approximately

$$E_{\text{pot}} \approx E_0 + \frac{1}{2}\int dx\, dy\, \left[\sigma\left(\frac{dz}{dx}\right)^2 + \sigma\left(\frac{dz}{dy}\right)^2 + \rho g z^2\right], \tag{4.3.43}$$

where E_0 is a constant, σ is the surface tension, ρ is the difference between the density of the liquid and that of the gas, and g is the gravitational acceleration.

a) For a periodic box of side length L, express the potential energy E_{pot} in terms of the Fourier coefficients $A(\boldsymbol{k})$ defined by

$$z(\boldsymbol{r}) = \frac{1}{L}\sum_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{r}} A(\boldsymbol{k}), \tag{4.3.44}$$

where $A(-\boldsymbol{k}) = A(\boldsymbol{k})^*$ and $\boldsymbol{k} = (k_x, k_y) = \frac{2\pi}{L}(n_x, n_y)$ (with n_x and n_y integers).

b) Due to thermal fluctuations,

$$\left\langle |A(\boldsymbol{k})|^2 \right\rangle = \frac{1}{ak^2 + b}, \tag{4.3.45}$$

as long as $|\boldsymbol{k}|$ is below a certain cutoff. What are the values of a and b at temperature T, in terms of the model's parameters (σ, ρ, T, L)?

c) Find an approximate expression for the r.m.s. width $W = \sqrt{\langle z(\boldsymbol{r})^2 \rangle}$, for wide containers, in terms of a, b, and the maximal value $\boldsymbol{k}_{\max}$ of $|\boldsymbol{k}|$. Assume also that $k_{\max}^2 \gg b/a$.

d) What determines $k_{\max}$?

Substituting (4.3.44) into the potential energy, we find

$$E_{\text{pot}} \approx E_0 + \frac{1}{2}\sum_{\boldsymbol{k}} \left(\sigma k^2 + \rho g\right) |A(\boldsymbol{k}|^2. \tag{4.3.46}$$

By the equipartition theorem, we have

$$\left\langle |A(\boldsymbol{k})|^2 \right\rangle = \frac{k_B T}{\sigma k^2 + \rho g}, \tag{4.3.47}$$

so $a = \frac{\sigma}{k_B T}$ and $b = \frac{\rho g}{k_B T}$.

The rms width will be

$$W = \sqrt{\langle z(\boldsymbol{r})^2 \rangle} = \frac{1}{L}\left(\sum_{\boldsymbol{k}} \left\langle |A(\boldsymbol{k})|^2 \right\rangle\right)^{1/2}. \tag{4.3.48}$$

Substituting the mean fluctuations, and approximating the sum by an integral, we find

$$W \approx \left(\frac{1}{2\pi}\int_0^{k_{\max}} \frac{k\,dk}{ak^2 + b}\right)^{1/2} = \sqrt{\frac{1}{4\pi a}} \log\left(1 + \frac{a}{b}k_{\max}^2\right). \tag{4.3.49}$$

Since we are assuming $k_{\max} \gg b/a$, the 1 in the logarithm is negligible.

The ultraviolet cutoff $k_{\max}$ will be determined by the length scale at which the water is no longer well-approximated by a continuum theory. This will be controlled by the atomic separation scale, so $k_{\max}$ should be on the order of inverse nanometers.

So much for equilibrium. What about systems out of equilibrium? For example, a sphere moving with some steady velocity v through water is clearly not in equilibrium, because its kinetic energy is much larger than the value prescribed by the equipartition theorem. Similarly, a sphere which is completely stationary in some water is not in equilibrium, because its kinetic energy is too low.

Both systems will equilibrate by the same means: interactions with water molecules. In the first case, collisions with water molecules will tend to slow down the sphere. In the second case, the same collisions will haphazardly accelerate the sphere in different directions, causing its velocity to fluctuate about its mean value of zero.

Macroscopically, however, these two processes look very different. We would describe the first process as dissipation: the useful mechanical energy of the sphere is dissipated into the heat of the water. The second process is fluctuation: a stationary sphere jostles about randomly. Since both of these macroscopic processes have the same microscopic origin, they are related. This is the content of the fluctuation-dissipation theorem.

To understand this relationship, suppose we have a particle of mass m which feels a drag force with coefficient γ, and also a stochastic force $R(t)$ due to thermal fluctuations:

$$m\ddot{x} = -\gamma\dot{x} + R(t). \tag{4.3.50}$$

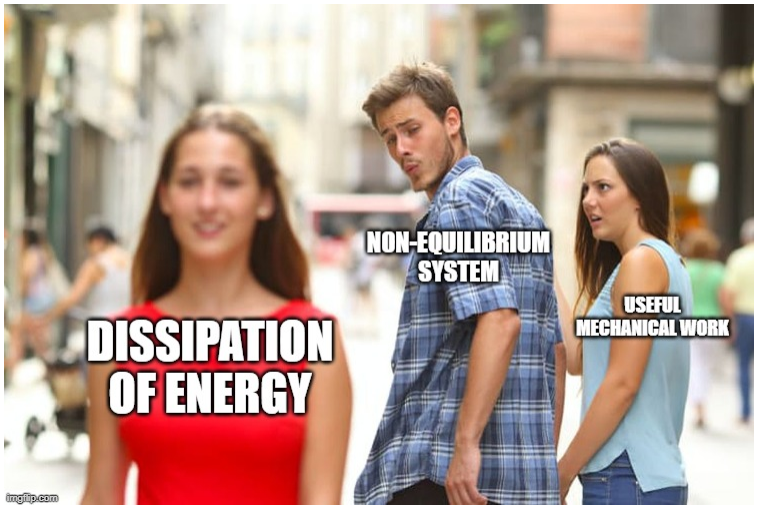


Figure 4.6

We can solve this equation via a Green's function:

$$\dot{x}(t) = \dot{x}(0)e^{-\gamma t/m} + \frac{1}{m}\int_{-\infty}^{t} R(s)e^{-\gamma(t-s)/m}\,ds. \tag{4.3.51}$$

From the equipartition theorem, we know that $\langle \dot{x}(t)^2 \rangle = \frac{k_B T}{m}$. Equilibrium will be achieved at late times, so

$$\frac{k_B T}{m} = \frac{1}{m^2}\int_{-\infty}^{\infty} ds_1 \int_{-\infty}^{\infty} ds_2\, \langle R(s_1)R(s_2)\rangle e^{-\frac{\gamma}{m}(2t-s_1-s_2)}. \tag{4.3.52}$$

We can simplify this by making the assumption that the statistics of the thermal fluctuations have time translation invariance, so $\langle R(s_1)R(s_2)\rangle = \langle R(0)R(s_2-s_1)\rangle$. We can then define $a = t - s_1$ and $b = s_2 - s_1$, and the integral becomes

$$\frac{k_B T}{m} = \frac{1}{m^2}\int_0^{\infty} da \int_{-\infty}^{\infty} db\, \langle R(0)R(b)\rangle e^{-\frac{\gamma}{m}(2a-b)} = \frac{1}{2\gamma m}\int_{-\infty}^{\infty} db\, \langle R(0)R(b)\rangle e^{\frac{\gamma b}{m}}. \tag{4.3.53}$$

We make the additional assumption that $\langle R(0)R(b)\rangle$ is only supported in a small interval around $b = 0$; if the stochastic force were correlated on large time scales, it wouldn't be the kind of random thermal force we're looking for. We can then ignore the exponential in the integrand, and we find

$$2\gamma k_B T = \int d\tau \langle R(0)R(\tau)\rangle. \tag{4.3.54}$$

This is the fluctuation-dissipation theorem: on the left we have the coefficient which controls the rate of energy dissipation, and on the right we have a quantity which measures the extent of thermal fluctuations.

Problem 4.15 (M00T2)
A solid spherical particle of radius b and mass M is suspended in a fluid, and is seen, using an optical microscope, to undergo Brownian motion. You are asked to show that a measurement of the mean-square displacement $\langle |\boldsymbol{r}(t_1) - \boldsymbol{r}(t_2)|^2 \rangle$ can be used to determine Boltzmann's constant.

Assume the densities of the solid and fluid are identical, so buoyancy can be ignored. The cause of the Brownian motion is a rapidly fluctuating force due to collisions with the molecules of the fluid. The force has mean zero, $\langle \boldsymbol{F}(t) \rangle = 0$, and two-time correlation of the form

$$\langle \boldsymbol{F}(t) \cdot \boldsymbol{F}(t') \rangle = C\delta(t - t'). \tag{4.3.55}$$

The fluid has viscosity η and the system is isothermal at temperature T. The equation of motion of the particle is

$$M\frac{d^2\boldsymbol{r}}{dt^2} + 6\pi b\eta \frac{d\boldsymbol{r}}{dt} = \boldsymbol{F}(t). \tag{4.3.56}$$

a) Express the velocity at time t as an integral involving the past forces, $\{\boldsymbol{F}(t')\}_{t'<t}$.

b) Find the coefficient C of the force-correlation at temperature T, as a function of T and the other constants mentioned above.

c) Describe the rate of growth of the mean square displacement $\langle |\boldsymbol{r}(t_1) - \boldsymbol{r}(t_2)|^2 \rangle$, and explain how its measurement can be used to determine Boltzmann's constant k_B.

The velocity can be expressed in terms of the Green's function,

$$\dot{\boldsymbol{r}}(t) = \frac{1}{M}\int_{-\infty}^{t} e^{-\frac{6\pi b\eta}{M}(t-t')}\boldsymbol{F}(t')\,dt'. \tag{4.3.57}$$

By the fluctuation-dissipation theorem, we have

$$C = 12\pi b\eta k_B T. \tag{4.3.58}$$

Man, I love hitting small nails with big hammers.

To find the mean square displacement, we integrate the velocity to find

$$\boldsymbol{r}(t) - \boldsymbol{r}(0) = \frac{1}{M}\int_0^t ds \int_{-\infty}^{s} e^{-\frac{6\pi b\eta}{M}(s-r)}\boldsymbol{F}(r)\,dr. \tag{4.3.59}$$

Thus,

$$\langle |\boldsymbol{r}(t) - \boldsymbol{r}(0)|^2 \rangle = \frac{1}{M^2}\int_0^t ds_1 \int_0^t ds_2 \int_{-\infty}^{s_1} dr_1 \int_{-\infty}^{s_2} dr_2\, e^{-\frac{6\pi b\eta}{M}(s_1+s_2-r_1-r_2)}\langle \boldsymbol{F}(r_1)\cdot\boldsymbol{F}(r_2)\rangle. \tag{4.3.60}$$

Thankfully the autocorrelation is as simple as it gets in this case, so this reduces down to

$$\langle |\boldsymbol{r}(t) - \boldsymbol{r}(0)|^2 \rangle = \frac{12\pi b\eta k_B T}{M^2}\int_0^t ds_1 \int_0^t ds_2 \int_{-\infty}^{\min(s_1,s_2)} dr\, e^{-\frac{6\pi b\eta}{M}(s_1+s_2-2r)}. \tag{4.3.61}$$

The remaining integrals are straightforward, and we find that for $t \gg \frac{M}{6\pi b\eta}$,

$$\langle |\boldsymbol{r}(t) - \boldsymbol{r}(0)|^2 \rangle = \frac{k_B T}{6\pi b\eta}t. \tag{4.3.62}$$

Thus, by measuring the average rate of increase of the mean square displacement, we can extract k_B.

4.4 Interacting Systems

Ideal gases are easy because they consist of noninteracting particles. What happens if the particles have some interactions? We will show here that the ideal gas equation of state is modified by corrections at higher powers of the density, a relation called the virial expansion. Furthermore, by computing the lowest-order correction term in this expansion, we will derive the van der Waals equation of state.

We start by computing the correction to the partition function due to a potential term in the Hamiltonian. The partition function will have an additional factor of $e^{-\beta\mathcal{V}}$ in the integrand, where $\mathcal{V}$ is the potential energy, so we can write it as

$$Z = Z_0 \left\langle e^{-\beta\mathcal{V}} \right\rangle_0 , \tag{4.4.1}$$

where Z_0 is the noninteracting partition function and $\langle\cdot\rangle_0$ denotes the expectation value with respect to the noninteracting distribution. The free energy is then

$$F = F_0 - k_B T \log \left\langle e^{-\beta\mathcal{V}} \right\rangle_0 . \tag{4.4.2}$$

We can expand the second term as a power series in β. The coefficients of this expansion are known in statistics as the cumulants of the random variable $\mathcal{V}$. We denote the nth cumulant by $\kappa_n(\mathcal{V})$, so

$$F = F_0 - \beta^{-1} \sum_{n=1}^{\infty} \frac{(-\beta)^n}{n!} \kappa_n(\mathcal{V}). \tag{4.4.3}$$

The first two cumulants are

$$\kappa_1(\mathcal{V}) = \langle \mathcal{V} \rangle, \qquad \kappa_2(\mathcal{V}) = \langle \mathcal{V}^2 \rangle - \langle \mathcal{V} \rangle^2. \tag{4.4.4}$$

We now assume the potential is a sum of identical pairwise interactions,

$$\mathcal{V} = \sum_{i<j} \mathcal{V}(\boldsymbol{x}_i - \boldsymbol{x}_j). \tag{4.4.5}$$

For the noninteracting gas, all the $\boldsymbol{x}_i$ are uniformly distributed over a volume V, so

$$\kappa_1(\mathcal{V}) = \sum_{i<j} \int \frac{d\boldsymbol{x}_i \, d\boldsymbol{x}_j}{V^2} \mathcal{V}(\boldsymbol{x}_i - \boldsymbol{x}_j) = \frac{N(N-1)}{2V} \int d\boldsymbol{x} \, \mathcal{V}(\boldsymbol{x}). \tag{4.4.6}$$

For the second cumulant, we have

$$\kappa_2(\mathcal{V}) = \sum_{i<j,k<l} \left(\langle \mathcal{V}(\boldsymbol{x}_i - \boldsymbol{x}_j) \mathcal{V}(\boldsymbol{x}_k - \boldsymbol{x}_l) \rangle - \langle \mathcal{V}(\boldsymbol{x}_i - \boldsymbol{x}_j) \rangle \langle \mathcal{V}(\boldsymbol{x}_k - \boldsymbol{x}_l) \rangle \right). \tag{4.4.7}$$

There are several simplifications to be made here. If the four indices are all different, then $\mathcal{V}(\boldsymbol{x}_i - \boldsymbol{x}_j)$ and $\mathcal{V}(\boldsymbol{x}_k - \boldsymbol{x}_l)$ are independent random variables and so the first term factors, and so these terms all vanish. Similarly, if only one index is shared, one can show that the term vanishes. Thus, we have

$$\kappa_2(\mathcal{V}) = \frac{N(N-1)}{2V} \left(\int d\boldsymbol{x} \, \mathcal{V}(\boldsymbol{x})^2 - \frac{1}{V} \left(\int d\boldsymbol{x} \, \mathcal{V}(\boldsymbol{x}) \right)^2 \right). \tag{4.4.8}$$

As long as the range of the potential is much less than the total volume, we can ignore the second term.

Similar simplifications occur at all orders. In general, the contributions to this expansion can be written as one-particle irreducible diagrams, much like Feynman diagrams, but here we are only interested in contributions at order N^2. By analogy to κ_2 we have

$$\kappa_p(V) = \frac{N(N-1)}{2V} \int d\boldsymbol{x}\, \mathcal{V}(\boldsymbol{x})^p + \mathcal{O}\left(\frac{N^3}{V^2}\right). \tag{4.4.9}$$

Thus,

$$F = F_0 - \frac{N(N-1)}{2\beta V} \sum_{p=1}^{\infty} \frac{(-\beta)^p}{p!} \int d\boldsymbol{x}\, \mathcal{V}(\boldsymbol{x})^p + \mathcal{O}\left(\frac{N^3}{V^2}\right). \tag{4.4.10}$$

The sum is just an exponential, and so we find

$$F = F_0 - \frac{N^2}{2\beta V} \int d\boldsymbol{x}\, \left(\exp(-\beta\mathcal{V}) - 1\right) + \mathcal{O}\left(\frac{N^3}{V^2}\right). \tag{4.4.11}$$

Our last job is to compute the integral. This is where the van der Waals point comes in: we assume the potential is a van der Waals attraction at large distances, $\mathcal{V}(r) \sim r^{-6}$. At short distances we want strong repulsion. This is sometimes modeled using a Lennard-Jones potential, but that sounds hard, so instead we use

$$\mathcal{V}(r) = \begin{cases} -u_0(r_0/r)^6 & r > r_0, \\ \infty & r < r_0 \end{cases}. \tag{4.4.12}$$

The parameter r_0 sets the size of the particles (the particle radius is $r_0/2$), and u_0 sets the strength of their interactions. Integrating, we find

$$\int d\boldsymbol{x}\, \left(\exp(-\beta\mathcal{V}) - 1\right) = -\frac{4\pi r_0^3}{3}\left(1 - \beta u_0\right). \tag{4.4.13}$$

Substituting this, we have

$$F = F_0 - \frac{N^2}{\beta V}\frac{2\pi r_0^3}{3}\left(1 - \beta u_0\right) + \mathcal{O}\left(\frac{N^3}{V^2}\right). \tag{4.4.14}$$

At last we are ready to compute the equation of state. Taking the derivative with respect to volume, we find

$$p = \frac{N k_B T}{V}\left(1 + \frac{N}{V}\frac{2\pi r_0^3}{3}\left(1 - \beta u_0\right)\right). \tag{4.4.15}$$

Rearranging, we find

$$\left(p + \frac{N^2}{V^2}\frac{2\pi r_0^3}{3}u_0\right) = \frac{N k_B T}{V}\left(1 + \frac{N}{V}\frac{2\pi r_0^3}{3}\right) \approx \frac{N k_B T}{V}\left(1 - \frac{N}{V}\frac{2\pi r_0^3}{3}\right)^{-1}, \tag{4.4.16}$$

where we have assumed that the total volume occupied by gas particles is much less than V. Finally we find

$$\left(p + \frac{aN^2}{V^2}\right)(V - Nb) = N k_B T, \tag{4.4.17}$$

where $b = \frac{2\pi r_0^3}{3}$ and $a = u_0 b$. This is the van der Waals equation of state.

Problem 4.16 (M08T2)
The van der Waals equation of state is

$$P = \frac{Nk_BT}{V - Nb} - a\frac{N^2}{V^2} \tag{4.4.18}$$

for the pressure P of a fluid N interacting atoms in a volume V at temperature T. This models the liquid-gas phase transition and its critical point.

a) Briefly explain the physics of each of the two above corrections to the ideal gas equation of state (corresponding to the parameters b and a).

b) Calculate the parameters at the critical point: the critical pressure P_c, critical temperature T_c, and the critical density $n_c = (N/V)_c$.

As we have seen, b measures the volume of the gas particles and a measures the strength of their interactions.

The critical point lies on the boundary of the unstable region of the van der Waals gas. When a gas is overpressured, it should expand until its pressure is low enough; this requires

$$\left(\frac{\partial p}{\partial V}\right)_T < 0. \tag{4.4.19}$$

The van der Waals gas behaves like an ideal gas at large volumes and pressures, but there is a region at small volumes and pressures where this condition is violated. Its boundary is given by the solutions to $\left(\frac{\partial p}{\partial V}\right)_T = 0$, or

$$-\frac{Nk_BT}{(V - Nb)^2} + \frac{2aN^2}{V^3} = 0. \tag{4.4.20}$$

For most values of T there will be two solutions; for the critical temperature there is only one solution. In the case of a double root, the second derivative also vanishes, so

$$\frac{2Nk_BT}{(V - Nb)^3} - \frac{6aN^2}{V^4} = 0. \tag{4.4.21}$$

Combining these, we find

$$\frac{6aN^2(V - Nb)}{V^4} = \frac{4aN^2}{V^3} \implies V_c = 3Nb. \tag{4.4.22}$$

Put another way, $(n_c) = \frac{1}{3b}$. Using the other equations we find

$$T_c = \frac{8}{27}\frac{a}{bk_B}, \qquad p_c = \frac{a}{27b^2}. \tag{4.4.23}$$

Another way to deal with interactions is to make the zeroth-order approximation that the interaction Hamiltonian can be replaced by a coupling to an external field, where the strength of the external field is determined by a self-consistency condition. This is called the mean field approximation. It is best illustrated by example.

Problem 4.17 (M07T3)
The Ising model on a 3-dimensional square lattice with spin-1/2 particles is defined by the Hamiltonian

$$H = -J \sum_{\boldsymbol{i},\boldsymbol{j}} \sigma_{\boldsymbol{i}} \sigma_{\boldsymbol{j}} - B \sum_{\boldsymbol{i}} \sigma_{\boldsymbol{i}} \tag{4.4.24}$$

where $J > 0$, $\boldsymbol{i}$ labels sites of the 3-dimensional lattice, $\boldsymbol{j}$ runs over nearest neighbor sites in 3 dimensions and $\sigma_{\boldsymbol{i}}$ is equal to $+1$ or -1.

The Ising model is often solved using the mean field approximation, consisting of replacing the spin interaction Hamiltonian by the mean field interaction

$$H_m = -M \sum_{\boldsymbol{i}} \sigma_{\boldsymbol{i}} - B \sum_{\boldsymbol{i}} \sigma_{\boldsymbol{i}} \tag{4.4.25}$$

where M is a parameter fixed by the self-consistency condition to be $M = 6J\langle\sigma_{\boldsymbol{i}}\rangle$.

a) For the Hamiltonian H_m calculate the free energy, entropy, and $\langle\sigma_{\boldsymbol{i}}\rangle$ at temperature T.

b) Show that for $B = 0$ at low temperature a self-consistent solution with $\langle\sigma_{\boldsymbol{i}}\rangle \neq 0$ has a lower free energy than a solution with $\langle\sigma_{\boldsymbol{i}}\rangle = 0$.

c) Find the critical temperature T_c above which the spontaneous magnetization vanishes at zero external field B.

d) How can one build a refrigerator using the spins as the working substance? Describe qualitatively how one can efficiently cool a substance by manipulating the spin degrees of freedom which obey this mean field theory.

The partition function is given by

$$Z = \sum_{n=0}^{N} \binom{N}{n} e\beta(M+B) \cdot (2n - N) = e^{-\beta N(M+B)} \left(1 + e^{2\beta(M+B)}\right)^N, \tag{4.4.26}$$

so the free energy is

$$F = N(M+B) - Nk_BT \log\left(1 + e^{2\beta(M+B)}\right). \tag{4.4.27}$$

The entropy is

$$S = -\frac{\partial F}{\partial T} = Nk_B \log\left(1 + e^{2\beta(M+B)}\right) - \frac{2N(M+B)/T}{1 + e^{-2\beta(M+B)}}. \tag{4.4.28}$$

The average spin is

$$\langle\sigma_{\boldsymbol{i}}\rangle = -1 + \frac{1}{N(M+B)} \frac{\partial(\log Z)}{\partial\beta} = \tanh(\beta(M+B)). \tag{4.4.29}$$

The self-consistency condition thus requires

$$M = 6J \tanh(\beta(M + B)). \tag{4.4.30}$$

At $B = 0$, we have

$$F = NM - Nk_BT \log\left(1 + e^{2\beta M}\right) = 6JN\langle\sigma_{\boldsymbol{i}}\rangle - Nk_BT \log\left(1 + e^{12\beta J\langle\sigma_{\boldsymbol{i}}\rangle}\right). \tag{4.4.31}$$

Taking a derivative, we find

$$\frac{dF}{d(\langle\sigma_{\boldsymbol{i}}\rangle)} = 6JN - \frac{12JN}{1 + e^{-12\beta J\langle\sigma_{\boldsymbol{i}}\rangle}}. \tag{4.4.32}$$

Evaluating at $\langle\sigma_{\boldsymbol{i}}\rangle = 0$ gives 0. Taking another derivative we have

$$\left.\frac{d^2F}{d(\langle\sigma_{\boldsymbol{i}}\rangle)^2}\right|_{\langle\sigma_{\boldsymbol{i}}\rangle=0} = -36\beta J^2 N, \tag{4.4.33}$$

so indeed $\langle\sigma_{\boldsymbol{i}}\rangle = 0$ is a local maximum of free energy and the system will tend to magnetize, provided a self-consistent nonzero magnetization M exists.

In order to have a solution to (4.4.30) for $B = 0$ and $M \neq 0$, we need

$$6J\beta > 1 \implies T < T_c \equiv \frac{6J}{k_B}. \tag{4.4.34}$$

We could build a weirdass refrigerator out of this system by taking advantage of the critical temperature. We start with the spins magnetized, and transfer heat from some other system to the spins until they reach T_c and demagnetize. Then we thermally isolate the spins and turn on an external field to magnetize the spins again, and repeat the cycle. This allows us to cool a system to T_c, but no more.

Problem 4.18 (M12T3)
Consider a classical one-dimensional magnet with Hamiltonian

$$H = -J\sum_{i=1}^{N} \boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1}, \tag{4.4.35}$$

where each $\boldsymbol{S}_i$ is a classical (3-component) vector spin of fixed length S.

a) Calculate $\langle \boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1}\rangle$ at equilibrium at temperature T.

b) Calculate the specific heat per spin $c(T)$ of this system in the limit $N \to \infty$.

c) Consider the $T \to 0$ limit of part b). Is this consistent with the behavior of $c(T)$ for a quantum ferromagnet ($J > 0$) of spin S with this same Hamiltonian? If not, estimate (roughly) and state the correct quantum behavior of $c(T)$ for small T, explaining your reasoning. Ferromagnetic spin waves in this model have a frequency that depends on wavenumber k as $\omega(k) \sim k^2$ for small k.

We may treat all the inner products $\boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1}$ as independent degrees of freedom. The partition function for a single one of these degrees of freedom is

$$Z_1 = \int_{-1}^{1} d(\cos\theta)\, e^{\beta J S^2 \cos\theta} = \frac{2\sinh(\beta J S^2)}{\beta J S^2}. \tag{4.4.36}$$

It follows that

$$\langle \boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1} \rangle = \frac{1}{J}\frac{\partial(\log Z)}{\partial\beta} = S^2 \coth(\beta J S^2) - \frac{1}{\beta J}. \tag{4.4.37}$$

The entropy is

$$S = k_B \left(1 + \log\left(\frac{2k_B T}{JS^2}\sinh\frac{JS^2}{k_B T}\right)\right) - \frac{JS^2}{T}\coth\left(\frac{JS^2}{k_B T}\right), \tag{4.4.38}$$

so

$$c(T) = T\frac{\partial S}{\partial T} = k_B - \frac{J^2 S^4}{k_B T^2}\,\mathrm{csch}^2\left(\frac{JS^2}{k_B T}\right). \tag{4.4.39}$$

As $T \to 0$, we have $c(T) \to k_B$ for the classical system. For a quantum ferromagnet, there is some minimum excitation energy which will be frozen out at low enough temperatures, and so $c(T) \to 0$.

Problem 4.19 (J12T2)
Consider N classical two-state spins $S_i = \pm 1$, with Hamiltonian

$$H = -\frac{J}{N}\sum_{i=1}^{N}\sum_{j=1}^{i-1} S_i S_j - h\sum_{i}^{N} S_i. \tag{4.4.40}$$

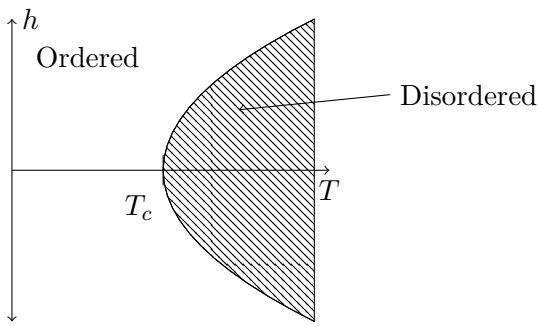
This is an "infinite-range" model where all spins interact with all others: the spin-spin coupling (J/N) is defined so the energy J remains a positive constant in the thermodynamic limit $N \to \infty$. You should work in this limit. The external magnetic field h can be of either sign or zero.

a) Sketch the equilibrium phase diagram of this system vs. temperature T and field h, showing all phase transitions and critical points that occur as one varies T and/or h.

b) Calculate the critical temperature T_c.

c) At the critical temperature $T = T_c$, calculate the magnetization $m = \langle S_i \rangle$ as a function of the field h in the limit where $|h|$ is small but nonzero.

We know there will be a critical temperature T_c above which the system is disordered at $h = 0$. For $h \neq 0$, the ordered phase can survive at higher temperatures. Thus, the phase diagram will look roughly as follows.



In the thermodynamic limit, the mean field approximation is exact in this case, and so we have

$$H = -(M + h) \sum_{i}^{N} S_i, \tag{4.4.41}$$

where $M = J\langle S_i \rangle$. Calculating this expectation, we find

$$M = J \tanh(\beta(M + h)). \tag{4.4.42}$$

At $h = 0$, a nonzero solution to this equation exists only if $T < T_c \equiv \frac{J}{k_B}$.

At the critical point we have

$$M = J \tanh\left(\frac{M + h}{J}\right) = M + h - \frac{(M + h)^3}{3J^2} + \mathcal{O}((M + h)^5), \tag{4.4.43}$$

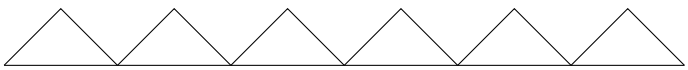
and solving this gives

$$M = (3J^2 h)^{1/3} - h. \tag{4.4.44}$$

Problem 4.20 (J01T3)
Mean field treatments of ferromagnetic systems are quite useful in general, but for certain "frustrated" antiferromagnets this may not be the case. Consider the antiferromagnetic Ising model on the triangular chain shown below, with Hamiltonian

$$H = J \sum_{\langle ij \rangle} s_i s_j, \tag{4.4.45}$$

where $J > 0$, the spins s_i have values ± 1, and the sum runs over all nearest neighbor pairs on the lattice (vertices joined by links in the figure).



a) Consider a system of N triangles at temperature T. Assuming that all the top row spins have zero thermal expectation values, write down a mean field theory for the spins. What is the transition temperature?

b) Consider a system consisting of a single triangle at $T = 0$. How many ground states (minimum energy configurations) does it have?

c) Consider a system of N triangles at $T = 0$. How many ground states does the chain have, supposing free boundary conditions?

d) Calculate the correlation $\langle s_i s_j \rangle$, averaged over the ground states for two spins on the bottom row (N triangles). Is this consistent with the answer in part a)?

Since the system is antiferromagnetic, we expect the spins on the bottom row to alternate. Let $M = 2J\langle\sigma_{2i}\rangle = -2J\langle\sigma_{2i+1}\rangle$, so that

$$H = M \sum_i (\sigma_{2i+1} - \sigma_{2i}) \,. \tag{4.4.46}$$

Since $M = J\langle\sigma_{2i} - \sigma_{2i+1}\rangle$, self-consistency fixes

$$M = 2J \tanh(\beta M), \tag{4.4.47}$$

which has nonzero solutions only for $T < T_c \equiv \frac{2J}{k_B}$.

For a single triangle, the best we can do is to have one spin up and two spins down, or vice versa; this gives six total ground states.

For a chain of N triangles, each one will have a single pair of spins which are equal. We have three ways of choosing this pair for each triangle, so there are $2 \cdot 3^N$ ground states.

We have

$$\langle s_i s_j \rangle = \langle s_i s_{i+1}^2 \cdots s_{j-1}^2 s_j \rangle = \langle s_i s_{i+1} \rangle \cdots \langle s_{j-1} s_j \rangle, \tag{4.4.48}$$

since the different triangles are independent. Neighboring spins on the bottom row have a $\frac{2}{3}$ chance of being different, so $\langle s_i s_{i+1} \rangle = -\frac{1}{3}$, and hence

$$\langle s_i s_j \rangle = \left(-\frac{1}{3}\right)^{|i-j|} . \tag{4.4.49}$$

This is not consistent with the mean field result, for which this expectation is $\pm \tanh(\beta M)$ for the self-consistent value of M (the sign depending on parity), independent of i and j.

4.5 Quantum Statistical Mechanics

Quantum mechanics is a probabilistic mess. So is statistical mechanics. Can you imagine what their kids would look like? Gross.

The ugly baby looks best when dressed in density matrices. We represent a pure state (a system

for which we know the quantum state with 100% certainty) by the outer product

$$\rho_\psi = |\psi\rangle \langle\psi| . \tag{4.5.1}$$

We then represent a statistical ensemble by a weighted sum of pure states,

$$\rho = \sum P(\psi) |\psi\rangle \langle\psi| . \tag{4.5.2}$$

Clearly every density matrix satisfies $\operatorname{tr}\rho = 1$. To determine whether a given density matrix represents a pure state, we can evaluate $\operatorname{tr}\rho^2$. We have

$$\operatorname{tr}\rho^2 = \sum P(\psi_i)P(\psi_j)|\langle\psi_i|\psi_j\rangle|^2 \leq \left(\sum P(\psi_i)\right)^2 = 1. \tag{4.5.3}$$

Equality holds only when $|\langle\psi_i|\psi_j\rangle| = 1$ for all i, j, which is clearly only possible if all the ψ_i are same up to a phase, in which case the density matrix can be written as $|\psi\rangle\langle\psi|$. Conversely, when $\operatorname{tr}\rho^2 < 1$, ρ represents a mixed state. More generally, the expectation value of some observable A is given by

$$\langle A\rangle = \sum P(\psi) \langle\psi|A|\psi\rangle = \operatorname{tr}(\rho A). \tag{4.5.4}$$

Problem 4.21 (J05Q1)
Two spin 1/2 particles interact via the Hamiltonian

$$H = -J\boldsymbol{S}_A \cdot \boldsymbol{S}_B. \tag{4.5.5}$$

At time $t = 0$, spin A points in the positive z-direction and spin B points in the negative z-direction. Compute the density matrix of spin A at time t. At which time does it describe a pure state, that is, at which time does the entanglement between the two spins vanish?

The Hamiltonian can be written as

$$H = \frac{J}{2}\left(S_A^2 + S_B^2 - (S_A + S_B)^2\right) = \frac{J}{2}\left(\frac{3}{2} - (S_A + S_B)^2\right). \tag{4.5.6}$$

Thus, the singlet state has energy $\frac{3J}{4}$ and the triplet states all have energy $-\frac{J}{4}$. The initial state is

$$|\uparrow\downarrow\rangle = \frac{1}{\sqrt{2}}\left(|0,0\rangle + |1,0\rangle\right), \tag{4.5.7}$$

so the state at time t is

$$\begin{aligned} |\psi(t)\rangle &= \frac{1}{\sqrt{2}}\left(e^{-3iJt/4\hbar}|0,0\rangle + e^{iJt/4\hbar}|1,0\rangle\right) \\ &= e^{-iJt/4\hbar}\left(\cos\left(\frac{Jt}{2\hbar}\right)|\uparrow\downarrow\rangle + i\sin\left(\frac{Jt}{2\hbar}\right)|\downarrow\uparrow\rangle\right). \end{aligned} \tag{4.5.8}$$

The density matrix is thus

$$\rho = \cos^2\left(\frac{Jt}{2\hbar}\right)|\uparrow\downarrow\rangle\langle\uparrow\downarrow| - \sin^2\left(\frac{Jt}{2\hbar}\right)|\downarrow\uparrow\rangle\langle\downarrow\uparrow| + i\sin\left(\frac{Jt}{2\hbar}\right)\cos\left(\frac{Jt}{2\hbar}\right)\left(-|\uparrow\downarrow\rangle\langle\downarrow\uparrow| + |\downarrow\uparrow\rangle\langle\uparrow\downarrow|\right). \tag{4.5.9}$$

Tracing over the second spin, we find

$$\rho_A = \cos^2\left(\frac{Jt}{2\hbar}\right)|\uparrow\rangle\langle\uparrow| - \sin^2\left(\frac{Jt}{2\hbar}\right)|\downarrow\rangle\langle\downarrow|. \tag{4.5.10}$$

This only describes a pure state when it is equal to either $|\uparrow\rangle\langle\uparrow|$ or $|\downarrow\rangle\langle\downarrow|$, i.e., when

$$t = \frac{n\pi\hbar}{J}, \qquad n \in \mathbb{Z}. \tag{4.5.11}$$

We can express the ensembles we have used classically as density matrices. The most important example is the canonical ensemble, for which we have

$$\rho = \frac{\sum e^{-\beta E_n}|\psi_n\rangle\langle\psi_n|}{Z}, \tag{4.5.12}$$

where the partition function Z fixes $\operatorname{tr}\rho = 1$. This means it is given by

$$Z = \operatorname{tr}\left(e^{-\beta H}\right). \tag{4.5.13}$$

Problem 4.22 (J98T3)
A sample, comprised of N independent spins ($s = 1/2$), sits in an external magnetic field H. Its Hamiltonian is given by

$$H = -g\mu_B \sum_n \boldsymbol{s}_n \cdot \boldsymbol{H}, \tag{4.5.14}$$

where $g = 2$ and μ_B is the Bohr magneton.

a) Write down the partition function Z.

b) Calculate the sample's entropy $S(H,T)$, and make a rough sketch of S versus the temperature T for a fixed field H_1. Show that S is a function of only one quantity x instead of two (T and H). How is x related to T and H? What is the T dependence of S in the low temperature limit?

c) The sample is initially connected to a heat bath at temperature T_0, with the field at H_1. The external field is increased slowly from H_1 to H_2 in an isothermal process. Calculate the heat Q exchanged with the bath. Which way does the heat flow? (Sketch the curve for S vs. T for a larger field.)

d) When the field reaches H_2, the link to the heat bath is removed. The field is then slowly reduced back to H_1 in an adiabatic process. Calculate the final temperature T_f of the sample.

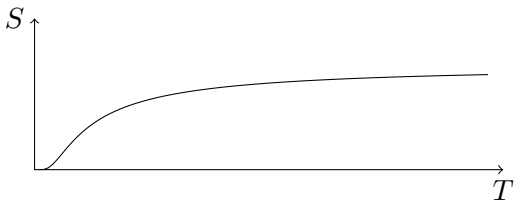
The partition function is

$$Z = \operatorname{tr}\left(e^{-\beta H}\right) = \left(2\cosh\left(\beta g\mu_B H\right)\right)^N. \tag{4.5.15}$$

The entropy is

$$S = -\frac{\partial F}{\partial T} = Nk_B \left(\log\left(2\cosh x\right) - x\tanh x\right), \tag{4.5.16}$$

where $x = \beta g\mu_B H$. (To be fair, the x thing is kind of vacuous...S is always a function of only one quantity, itself). The plot below shows how the entropy depends on temperature.



At low temperatures x is very large, and so

$$S \sim Nk_B x\left(1 - \tanh x\right) \sim Nk_B x e^{-2x}. \tag{4.5.17}$$

The heat transferred is

$$Q = \frac{S(H_2) - S(H_1)}{T} \tag{4.5.18}$$

$$= Nk_B \left(\log\frac{\cosh(\beta g\mu_B H_2)}{\cosh(\beta g\mu_B H_1)} + \beta g\mu_B \left(H_1\tanh(\beta g\mu_B H_1) - H_2\tanh(\beta g\mu_B H_2)\right)\right). \tag{4.5.19}$$

The entropy decreases as H is increased, so heat flows out of the system.

Since the entropy depends only on $\frac{H}{T}$, the final temperature is

$$T_f = T\frac{H_1}{H_2}. \tag{4.5.20}$$

Problem 4.23 (J04T1)
Consider N non-interacting quantized spins in a magnetic field $\boldsymbol{B} = B\hat{\boldsymbol{z}}$. The energy of the spins is $-BM_z$, where

$$M_z \equiv \mu\sum_{i=1}^{N} S_z^{(i)} \tag{4.5.21}$$

is the total magnetization. For each spin, $S_z^{(i)}$ takes only $2S+1$ values $-S, -S+1, \ldots, S-1, S$. Given the temperature of the system T:

a) Calculate the Gibbs partition function $Z(T, B)$;

b) Calculate the Gibbs free energy $G(T, B)$ and evaluate its asymptotic behavior at weak ($\mu BS \ll k_B T$) and strong ($\mu B \gg k_B T$) magnetic field;

c) Calculate the zero-field magnetic susceptibility

$$\chi = \left(\frac{\partial M_z}{\partial B}\right)_{B=0} \tag{4.5.22}$$

d) Calculate the magnetic susceptibility at strong fields $\mu B \gg k_B T$.

The Gibbs-ness comes from thinking of the magnetic field as an external force rather than a part of the system's Hamiltonian; the calculation is exactly the same as usual. The partition function is

$$Z = \left(e^{\beta\mu BS} + e^{\beta\mu B(S-1)} + \ldots + e^{-\beta\mu BS}\right)^N = \left(\frac{e^{\beta\mu BS} - e^{-\beta\mu B(S+1)}}{1 - e^{-\beta\mu B}}\right)^N. \tag{4.5.23}$$

The free energy is

$$G = -Nk_B T \log \frac{e^{\beta\mu BS} - e^{-\beta\mu B(S+1)}}{1 - e^{-\beta\mu B}}. \tag{4.5.24}$$

For weak fields, we have

$$G \to -Nk_B T \log(2S+1), \tag{4.5.25}$$

and for strong fields,

$$G \to -N\mu BS. \tag{4.5.26}$$

Since we are looking at the zero-field susceptibility, we can use the weak field partition function. Expanding to second order in the field, we have

$$\begin{aligned} Z &= \left((2S+1) + \frac{1}{6}S(S+1)(2S+1)(\beta\mu B)^2 + \mathcal{O}(B^3)\right)^N \\ &= (2S+1)^N + \frac{1}{6}NS(S+1)(2S+1)^N(\beta\mu B)^2 + \mathcal{O}(B^3). \end{aligned} \tag{4.5.27}$$

We then have

$$M_z = \frac{1}{B}\frac{\partial(\log Z)}{\partial\beta} = \frac{1}{3}NS(S+1)\beta\mu^2 B + \mathcal{O}(B^2), \tag{4.5.28}$$

and so

$$\chi = \frac{1}{3}NS(S+1)\beta\mu^2. \tag{4.5.29}$$

At strong fields, the partition function is $e^{\beta N\mu BS}$, so

$$M_z = \frac{1}{B}\frac{\partial(\log Z)}{\partial\beta} = N\mu S, \tag{4.5.30}$$

and $\chi = 0$.

These problems show that the mere fact of having a quantum system doesn't change things too much: we just use the energy eigenstates as the states, and proceed as before. The fun comes from identical particles. The state of a system of identical bosons must be symmetric under the exchange symmetry. For fermions, the state must be antisymmetric under exchange. These rules, known as Bose-Einstein statistics and Fermi-Dirac statistics respectively, lead to corrections to the

Figure 4.7

bulk thermodynamic properties of quantum systems.

Problem 4.24 (M03T3)
Consider a single free particle of mass m confined to a volume V. Let $Z_1(m)$ denote the quantum partition function for this system (where the partition sum is taken over the discrete energy levels of a particle of mass m in a box of volume V).

a) Show that $Z_1(m) \to V/\lambda^3$ with $\lambda = h/\sqrt{2\pi m k_B T}$ in the classical (or small $\hbar$) limit. Use this result to calculate the classical energy and heat capacity at fixed volume of the single particle system.

b) Identify the temperature at which this approximation breaks down.

c) Now consider a system consisting of two identical, non-interacting particles in the same box. Because of the effects of identical particle statistics, the classical expectation for the two-particle partition function $Z_2(m) = Z_1(m)^2$ is not quite correct. Show that the exact free boson and free fermion two-particle partition sums can in fact be expressed in a simple way in terms of the one-particle functions $Z_1(m)$ and $Z_1(m/2)$.

d) Using the classical approximation $Z_1(m) = V/\lambda^3$ derived in the first part of this problem, calculate the correction to the energy E and the heat capacity C due to Bose or Fermi statistics.

Assume the volume is a cube with side length $L = V^{1/3}$. The partition function is given by

$$Z_1(m) = \mathrm{tr}\left(e^{-\beta H}\right) = \sum_{n_x=1}^{\infty}\sum_{n_y=1}^{\infty}\sum_{n_z=1}^{\infty} \exp\left(-\frac{\pi^2\hbar^2}{2mk_BTL^2}\left(n_x^2+n_y^2+n_z^2\right)\right). \tag{4.5.31}$$

As $\hbar \to 0$ we can approximate the sums by integrals, and we find

$$Z_1(m) \approx \int_0^\infty \frac{\pi}{2} r^2\, dr\, \exp\left(-\frac{\pi^2\hbar^2}{2mk_BTV^2}r^2\right) = \frac{\pi^{3/2}}{8}\left(\frac{2mk_BTL^2}{\pi^2\hbar^2}\right)^{3/2} = \frac{V}{\lambda^3}. \tag{4.5.32}$$

The only approximation we made was in treating the sum as an integral. This only makes sense when

$$T \gg \frac{\pi^2\hbar^2}{2mk_BV^2}, \tag{4.5.33}$$

or equivalently when $\lambda^3 \ll V$.

If we square the partition function, we get two copies of each off-diagonal state, but for either bosons or fermions we only want one copy. For bosons we still want one copy of the diagonal states, and for fermions we want to exclude the diagonal states. The diagonal part of $Z_1(m)^2$ is $Z_1(m/2)$, so we have

$$Z_2(m) = \frac{1}{2}\left(Z_1(m)^2 \pm Z_1(m/2)\right), \tag{4.5.34}$$

where the positive sign is for bosons and the negative sign is for fermions.

In the classical limit this gives

$$Z_2(m) = \frac{1}{2}\left(\frac{V^2}{\lambda^6} \pm 2^{3/2}\frac{V}{\lambda^3}\right). \tag{4.5.35}$$

The energy is

$$E = -\frac{\partial(\log Z)}{\partial\beta} = 3k_BT\frac{\frac{V}{\lambda^3} \pm 2^{1/2}}{\frac{V}{\lambda^3} \pm 2^{3/2}} \approx 3k_BT\left(1 \mp \sqrt{2}\frac{\lambda^3}{V}\right). \tag{4.5.36}$$

The heat capacity is

$$C = -T\frac{\partial^2 F}{\partial^2 T} \approx 3k_B\left(1 \mp \sqrt{2}\frac{\lambda^3}{V}\right). \tag{4.5.37}$$

Naturally the corrections are of order $\frac{\lambda^3}{V}$, so the classical expressions hold for $\lambda^3 \ll V$.

Problem 4.25 (J05T2)
Consider $N \gg 1$ spinless noninteracting bosons contained in an isotropic three-dimensional harmonic well. In terms of the position $\boldsymbol{r}$ and the momentum $\boldsymbol{p}$, the single-particle Hamiltonian is

$$H = \frac{1}{2m}|\boldsymbol{p}|^2 + \frac{1}{2}m\omega_0^2|\boldsymbol{r}|^2, \tag{4.5.38}$$

where the particles have mass m and the oscillations in the potential have natural frequency ω_0. The resulting energy levels depend on the three quantum numbers

$$E = \hbar\omega_0(n_x + n_y + n_z + (3/2)), \tag{4.5.39}$$

where each $n_i = 0, 1, 2, \ldots$. This can also be represented as energy levels that depend on a single quantum number $n = 0, 1, 2, \ldots$; $\epsilon_n = \epsilon_0 + n\hbar\omega_0$, but with a degeneracy $g_n = (n+1)(n+2)/2$, and $\epsilon_0 = \frac{3}{2}\hbar\omega_0$.

a) What is the specific heat $c_N(T)$ per particle, at fixed particle number N, in the "classical limit" where $k_BT/\hbar\omega_0$ is so large that $N_0 \ll 1$, where N_0 is the mean number of particles in the $n = 0$ state.

b) Find $c_N(T)$ at low temperatures $k_BT \ll \hbar\omega_0$, including the leading behavior for nonzero temperatures.

c) Find the chemical potential, $\mu(T, N)$ in the "classical limit." Above what temperature scale is the "classical limit" reached?

d) Find $\mu(T, N)$ for low temperatures $k_B T \ll \hbar\omega_0$, including the leading behavior for non-zero temperatures.

e) Since the particles are bosons, $N_0(T, N)$ may be macroscopic (i.e. of order N) in a finite temperature range $T < T_{\rm BEC}(N, \hbar\omega_0)$. Obtain an expression for $T_{\rm BEC}$ in the large-N limit. You can express any numerical constants as dimensionless integrals which you must define, but need not evaluate.

The partition function for a single particle in a one-dimensional oscillator is $\sum e^{-\beta\epsilon_n} = \frac{e^{\hbar\omega/2}}{e^{\hbar\omega}-1}$, so the classical partition function for N particles in a three-dimensional oscillator is

$$Z_N = \frac{1}{N!}\left(\frac{e^{\beta\hbar\omega/2}}{e^{\beta\hbar\omega}-1}\right)^{3N}. \tag{4.5.40}$$

This gives

$$F = -\frac{3}{2}N\hbar\omega + 3Nk_BT\log\left(e^{\beta\hbar\omega}-1\right) + Nk_BT(\log N - 1). \tag{4.5.41}$$

The heat capacity per particle is

$$c_n(T) = -\frac{T}{N}\frac{\partial^2 F}{\partial^2 T} = 3k_B\frac{(\beta\hbar\omega)^2 e^{\beta\hbar\omega}}{(e^{\beta\hbar\omega}-1)^2}. \tag{4.5.42}$$

In the classical limit, this becomes

$$c_n(T) \to 3k_B. \tag{4.5.43}$$

For low temperatures, the classical partition function will not be precisely correct, because the factor of $\frac{1}{N!}$ does not properly account for Bose statistics. However, this will not affect the heat capacity, so we can just take the low temperature limit of the formula above to find

$$c_n(T) \to \frac{3\hbar^2\omega^2}{k_BT^2}e^{-\beta\hbar\omega}. \tag{4.5.44}$$

The chemical potential is

$$\mu = \frac{\partial F}{\partial N} = 3k_BT\log\left(e^{\beta\hbar\omega}-1\right) + k_BT\log N - \frac{3}{2}\hbar\omega. \tag{4.5.45}$$

In the classical limit, where $k_BT \gg \hbar\omega$,

$$\mu \to 3k_BT\log\frac{N^{1/3}\hbar\omega}{k_BT}. \tag{4.5.46}$$

At low temperatures for the chemical potential, we have to start worrying about Bose statistics, since nearly all the particles will share the ground state. In the grand canonical ensemble, we have

$$\mathcal{Z} = \sum_{n=0}^{\infty} e^{n\beta(\mu-\epsilon_0)} = \frac{1}{1-e^{\beta(\mu-\epsilon_0)}}, \tag{4.5.47}$$

and so the expected number of particles in the ground state is

$$\langle N_0 \rangle = \frac{1}{\beta} \frac{\partial (\log \mathcal{Z})}{\partial \mu} = \frac{1}{e^{-\beta(\mu - \epsilon_0)} - 1}. \tag{4.5.48}$$

Solving for the chemical potential, we find

$$\mu = \epsilon_0 + k_B T \log \left(1 - \frac{1}{N} \right) \approx \frac{3}{2} \hbar \omega + \frac{k_B T}{N}. \tag{4.5.49}$$

As $T \to 0$ we have $\mu \to \epsilon_0$, and so $\langle N_0 \rangle \to \infty$, so all the particles will be in the ground state. A Bose-Einstein condensate is formed when we can replicate this behavior at finite temperature and finite N. That is, we want $\mu = \epsilon_0$. If we trusted (4.5.45) in this regime – which we do not – this would mean

$$3\beta\hbar\omega \left(1 - \log \left(e^{\beta\hbar\omega} - 1 \right) \right) = \log N, \tag{4.5.50}$$

which gives

$$T = -\frac{\hbar\omega}{k_B \log(1 - N^{-1/3})} \sim \frac{\hbar\omega N^{1/3}}{k_B}. \tag{4.5.51}$$

We expect to find the same scaling in the final result, possibly with a different overall coefficient. Following a similar computation as in the previous part, but for all the states, we have

$$N = \sum_{n=0}^{\infty} \frac{g_n}{e^{-\beta(\mu - \epsilon_0 - n\hbar\omega)} - 1}. \tag{4.5.52}$$

Setting $\mu = \epsilon_0$, we have

$$N = \sum_{n=0}^{\infty} \frac{g_n}{e^{\beta n \hbar\omega} - 1} \approx \int_0^{\infty} \frac{dx}{\beta\hbar\omega} \frac{\frac{1}{2}\left(\frac{x}{\beta\hbar\omega} + 2 \right)\left(\frac{x}{\beta\hbar\omega} + 1 \right)}{e^x - 1} \approx \frac{1}{2(\beta\hbar\omega)^3} \int_0^{\infty} \frac{x^2\, dx}{e^x - 1}, \tag{4.5.53}$$

where we have assumed $\beta\hbar\omega \gg 1$. The integral is

$$\int_0^{\infty} \frac{x^2 e^{-x}\, dx}{1 - e^{-x}} = \int_0^{\infty} x^2 e^{-x} \left(1 + e^{-x} + e^{-2x} + \ldots \right) dx = \Gamma(3) \left(1 + \frac{1}{2^3} + \ldots \right) = 2\zeta(3), \tag{4.5.54}$$

so

$$T = \left(\frac{N}{\zeta(3)} \right)^{1/3} \frac{\hbar\omega}{k_B}, \tag{4.5.55}$$

a result of the form we expected.

4.6 Quantum Gases

We can give a unified treatment of Bose-Einstein and Fermi-Dirac statistics in the grand canonical ensemble. The grand partition function is

$$\mathcal{Z} = \sum_{N=0}^{\infty} e^{\beta N\mu} \sum_{\{n_i\}} e^{-\beta \sum n_i \epsilon_i}, \tag{4.6.1}$$

where the inner sum ranges over all allowed sets of occupation numbers with $\sum n_i = N$. For bosons any occupation number is allowed, but for fermions only $n_i = 0, 1$ are allowed. As usual with the grand partition function, we can rearrange the sum to find

$$\mathcal{Z} = \prod_i \left(\sum_{n_i} e^{\beta n_i(\mu - \epsilon_i)} \right). \tag{4.6.2}$$

For fermions the sum has only two terms, and for bosons it is an infinite geometric series. Letting $\eta = +1$ for bosons and $\eta = -1$ for fermions, we can express this as

$$\log \mathcal{Z} = -\eta \sum_i \log \left(1 - \eta e^{\beta(\mu - \epsilon_i)} \right). \tag{4.6.3}$$

Taking the derivative with respect to μ, we can find the expected occupation number of the ith state,

$$\langle N_i \rangle = \frac{1}{e^{\beta(\epsilon_i - \mu)} - \eta}. \tag{4.6.4}$$

Problem 4.26 (M06T3)
Consider a 3-dimensional gas of (spinless, non-relativistic) bosons at pressure P and temperature T. The bosons can be absorbed onto a (2-dimensional) surface layer, where they are bound with energy $-\epsilon_0 < 0$, but retain their translational degrees of freedom in 2 dimensions. The (ideal) 3D gas is in equilibrium with the (ideal) 2D adsorbed gas. Treating the 3D gas classically, but the 2D (absorbed) gas quantum mechanically, compute the surface density in the layer as a function of P and T.

The number of particles in the 3D gas can be written as

$$N_3 = V \int d^3\boldsymbol{k}\, e^{\beta(\mu - \hbar^2 k^2/2m)} = e^{\beta\mu} \left(\frac{2\pi m k_B T}{\hbar^2} \right)^3 \equiv e^{\beta\mu} \frac{V}{\lambda^3}, \tag{4.6.5}$$

so the chemical potential is

$$\mu = k_B T \log \frac{N_3 \lambda^3}{V} = k_B T \log \frac{P\lambda^3}{k_B T}. \tag{4.6.6}$$

Since the two species are in equilibrium, they have the same chemical potential. This allows us to determine the number of particles in the 2D gas, using the Bose-Einstein distribution:

$$N_2 = A \int \frac{d^2\boldsymbol{k}}{e^{\beta(\hbar^2 k^2/2m - \epsilon_0 - \mu)} - 1} = A \int \frac{2\pi k\, dk}{\frac{k_B T}{P\lambda^3} e^{-\beta\epsilon_0} e^{\lambda^2 k^2} - 1} = A \log \frac{k_B T}{k_B T - P\lambda^3 e^{\beta\epsilon_0}}. \tag{4.6.7}$$

Thus, the surface density is

$$\frac{N_2}{A} = \log \frac{k_B T}{k_B T - P\lambda^3 e^{\beta\epsilon_0}}. \tag{4.6.8}$$

Problem 4.27 (M08T3)
Assume that, to escape from a metal, an electron must impinge from the interior onto the surface with enough momentum to overcome the confining potential that holds the electrons in the metal. Also assume that all electrons with such a momentum do escape. Calculate the flux (number per area per time) of electrons escaping from a metal with work function ϕ (the barrier energy) at room temperature T. Treat the electrons as an ideal Fermi gas.

For simplicity, let the metal be a cube with side length L. We need to integrate over all wavenumbers with $\frac{\hbar^2 k^2}{2m_e} > \phi + \mu$. An electron with wavenumber $\boldsymbol{k}$ will escape in inverse time $\frac{1}{\tau} = \frac{2\hbar|\boldsymbol{k}|}{Lm_e}$. Thus, the flux is

$$F = \frac{1}{6L^2}\frac{L^3}{(2\pi)^3}\int_{\sqrt{2m_e(\phi+\mu)}/\hbar}^{\infty} \frac{4\pi k^2\,dk}{e^{\beta(\hbar^2k^2/2m_e-\mu)}+1}\frac{2\hbar k}{Lm_e} = \frac{\hbar}{6\pi^2 m_e}\int_{\sqrt{2m_e(\phi+\mu)}/\hbar}^{\infty} \frac{k^3\,dk}{e^{\beta(\hbar^2k^2/2m_e-\mu)}+1}. \tag{4.6.9}$$

Let $x = \beta\frac{\hbar^2k^2}{2m}$, so this becomes

$$F = \frac{2m_e}{3\pi^2\beta^2\hbar^3}\int_{\beta(\phi+\mu)}^{\infty} \frac{x\,dx}{e^{-\beta\mu}e^x+1}. \tag{4.6.10}$$

A typical metal at room temperature has $k_B T \ll E_F$, so we can ignore the 1 in the denominator and set $\mu = E_F$, and we find

$$F = \frac{2m_e}{3\pi^2\beta^2\hbar^3}\left(1+\beta(E_F+\phi)\right)e^{-\beta\phi} \approx \frac{2m_e}{3\pi^2\beta\hbar^3}(E_F+\phi)e^{-\beta\phi}. \tag{4.6.11}$$

Ideal quantum gases do not behave like ideal classical gases at low temperatures. We can parametrize the effect using a virial expansion. We start from the grand potential,

$$\mathcal{G} = \eta k_B T \sum_i \log\left(1-\eta e^{\beta(\mu-\epsilon_i)}\right). \tag{4.6.12}$$

Let's assume we have a gas in a volume V. Then we can replace the sum by an integral over wavenumbers, with a factor of $\frac{Vg}{(2\pi)^3}$ to get the number of states correct (where $g = 2s+1$ accounts for spin-s degeneracy). The energy is $\epsilon_{\boldsymbol{k}} = \frac{\hbar^2k^2}{2m}$. The grand potential is then

$$\mathcal{G} = \eta g k_B T V \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\log\left(1-\eta e^{\beta(\mu-\hbar^2k^2/2m)}\right). \tag{4.6.13}$$

The pressure is given by

$$p = -\frac{\partial\mathcal{G}}{\partial V} = -\eta g k_B T \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\log\left(1-\eta e^{\beta(\mu-\hbar^2k^2/2m)}\right). \tag{4.6.14}$$

This is kind of gross, but we can make it better. The integrand only depends on $|\boldsymbol{k}|$, so we can replace $d^3\boldsymbol{k} \to 4\pi k^2\, dk$. Integrating by parts, we find

$$p = -\eta g k_B T \int_0^\infty \frac{4\pi k^2\, dk}{(2\pi)^3} \log\left(1 - \eta e^{\beta(\mu - \hbar^2 k^2/2m)}\right) = \frac{g\hbar^2}{6m\pi^2} \int_0^\infty \frac{k^4\, dk}{e^{\beta(\hbar^2 k^2/2m - \mu)} - \eta}. \tag{4.6.15}$$

Now this is looking more like an integral over the number distribution. In fact, if we compute the energy we find

$$U = gV \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} \frac{\hbar^2 k^2}{2m} \frac{1}{e^{\beta(\hbar^2 k^2/2m - \mu)} - \eta} = \frac{g\hbar^2 V}{4m\pi^2} \int_0^\infty dk\, \frac{k^4\, dk}{e^{\beta(\hbar^2 k^2/2m - \mu)} - \eta}. \tag{4.6.16}$$

We thus have

$$U = \frac{3}{2} pV, \tag{4.6.17}$$

just as with the ideal monatomic gas.

To find the equation of state, we have to compare the pressure to the number of particles,

$$N = \frac{gV}{2m\pi^2} \int_0^\infty \frac{k^2\, dk}{e^{\beta(\hbar^2 k^2/2m - \mu)} - \eta}. \tag{4.6.18}$$

De-dimensionalizing both the number and the pressure, we find

$$p = k_B T \frac{g}{\lambda^3} \frac{4}{3\sqrt{\pi}} \int_0^\infty \frac{x^{3/2}\, dx}{z^{-1} e^x - \eta}, \tag{4.6.19}$$

$$N = \frac{gV}{\lambda^3} \frac{2}{\sqrt{\pi}} \int_0^\infty \frac{x^{1/2}\, dx}{z^{-1} e^x - \eta}, \tag{4.6.20}$$

where $z = e^{\beta\mu}$. Both of these integrals can be expanded in series in the usual way, so we find

$$p = k_B T \frac{g}{\lambda^3} \left(z + \eta \frac{z^2}{2^{5/2}} + \frac{z^3}{3^{5/2}} + \dots\right), \tag{4.6.21}$$

$$N = \frac{gV}{\lambda^3} \left(z + \eta \frac{z^2}{2^{3/2}} + \frac{z^3}{3^{3/2}} + \dots\right). \tag{4.6.22}$$

Now things are looking good. At order z we recover the ideal gas equation of state, and we can make corrections at each higher order:

$$pV = N k_B T \left(1 - \frac{\eta}{2^{5/2}} \frac{\lambda^3}{g} \frac{N}{V} + \mathcal{O}\left((N/V)^2\right)\right). \tag{4.6.23}$$

This is the virial expansion for a quantum gas. From the sign of the first correction term, we see that fermions experience an effective repulsion, while bosons experience an effective attraction, as we might have guessed from their statistics.

Problem 4.28 (J14T2)
Consider a Fermi gas of N non-interacting particles in d dimensions where each particle has kinetic energy K.E. $= a|\boldsymbol{p}|^\nu$. The Fermi gas is placed in a box of volume V. Here, a and ν are positive constants, and N is assumed to be very large.

a) The Fermi energy can be written approximately as $E_F \approx \gamma N^\lambda$ for some γ and λ. Determine the exponent λ in terms of d and ν.

b) How does the heat capacity scale with temperature and the number of particles at small temperatures? Give the answer in terms of λ.

c) For this Fermi gas at temperature $T > 0$ the pressure P is related to the total energy E through $P = \alpha E/V$. Find α in terms of ν and d.

d) For a relativistic Fermi gas in 3 dimensions $\nu = 1$. For this case derive $P = \alpha E/V$ also from the kinetic theory, with P expressed as the force per unit area exerted by the gas particles on the walls of the container.

The density of states is uniform in momentum space, and so $k_F \propto N^{1/d}$. We then have $E_F \propto N^{\nu/d}$, so $\lambda = \frac{\nu}{d}$.

The energy can be written as

$$E = \int_0^\infty \frac{\epsilon g(\epsilon)\, d\epsilon}{e^{\beta(\epsilon-\mu)}+1} = \int_0^\infty \frac{\epsilon^{1/\lambda}\, d\epsilon}{e^{\beta(\epsilon-\mu)}+1}. \tag{4.6.24}$$

We have shown formally in some other problem that the leading order correction to the $T = 0$ value enters at order T^2 and is proportional to the derivative of the numerator, so

$$C = \frac{\partial E}{\partial T} \propto E_F^{1/\lambda-1} T \propto N^{1-\lambda} T, \tag{4.6.25}$$

where we have taken $\mu = E_F$.

To find the pressure, we start from the grand potential. Ignoring unimportant overall factors, we have

$$\mathcal{G} = -k_B T V \int d^3\boldsymbol{k} \log\left(1 + e^{\beta(\mu - a(\hbar k)^\nu)}\right). \tag{4.6.26}$$

Differentiating with respect to V, and integrating by parts, we find

$$p = \frac{\nu a \hbar^\nu}{d} \int \frac{k^{d+\nu-1}\, dk}{e^{\beta(a(\hbar k)^\nu - \mu)}+1}. \tag{4.6.27}$$

The energy is given by

$$E = V \int \frac{a(\hbar k)^\nu k^{d-1}\, dk}{e^{\beta(a(\hbar k)^\nu - \mu)}+1}. \tag{4.6.28}$$

Thus, $\alpha = \frac{\nu}{d}$.

Imagine the relativistic Fermi gas consists of a single particle of momentum p in a volume $V = L^3$. Furthermore, imagine that its momentum is directed along the x axis (I know, it's a lot of imagination). Its velocity is $\frac{dE}{dp} = a$, so it imparts momentum $2p$ to a wall with a frequency $\frac{a}{L}$, giving an average force $\frac{2pa}{L}$, or a pressure $\frac{pa}{3L^3} = \frac{pa}{3V}$. The energy is just ap, so indeed we find $\alpha = \frac{1}{3} = \frac{\nu}{d}$.

Now we need to understand what happens when things get degenerate. Of course, the word "degenerate" has two meanings. One is the common meaning, for example, "look at that guy, what a degenerate!" I asked my mom what "degenerate" meant after someone said that to me, and she said it means someone cool that you'd like to hang out with. So that felt good. Anyway, in the context of quantum gases, "degenerate" means the temperature is so low and/or the density is so high that the bulk thermodynamic properties are controlled by quantum effects. For Bose gases, we have already solved problems where the chemical potential becomes equal to the ground state energy at finite temperatures, so that all the particles are in the ground state of the system. For Fermi gases, degeneracy means that the temperature is near zero, so the particles are mostly occupying the lowest N states. In either case, the virial expansion we derived becomes useless because it converges very slowly.

We will instead derive an expansion, known as the Sommerfeld expansion, which describes how the Fermi-Dirac distribution function behaves at low temperatures. When we de-dimensionalize the integral expressions for thermodynamic quantities, we always end up with integrals of the form

$$f_m^-(z) = \frac{1}{\Gamma(m)} \int_0^\infty \frac{x^{m-1}\,dx}{z^{-1}e^x + 1}, \tag{4.6.29}$$

where $z = e^{\beta\mu}$. At low temperatures we have $z \to \infty$, so we need an asymptotic expression for $f_m^-(z)$. The key idea is that for small T, the Fermi-Dirac distribution is nearly a step function, so the low-temperature corrections are dominated by the region around $\log z$. We can see this formally by integrating by parts, giving

$$f_m^-(z) = -\frac{1}{\Gamma(m+1)} \int_0^\infty x^m \frac{d}{dx}\left(\frac{1}{z^{-1}e^x + 1}\right) dx. \tag{4.6.30}$$

Letting $x = \log z + t$, this becomes

$$\begin{aligned} f_m^-(z) &= -\frac{1}{\Gamma(m+1)} \int_{-\log z}^\infty (\log z + t)^m \frac{d}{dt}\left(\frac{1}{e^t + 1}\right) dx \\ &= -\frac{(\log z)^m}{\Gamma(m+1)} \sum_{k=0}^\infty \binom{m}{k} (\log z)^{-k} \int_{-\log z}^\infty t^k \frac{d}{dt}\left(\frac{1}{e^t+1}\right) dt. \end{aligned} \tag{4.6.31}$$

Since z is large, we can pretend the integral extends all the way to $-\infty$. Since

$$\frac{d}{dt}\left(\frac{1}{e^t+1}\right) = \frac{d}{dt}\left(1 - \frac{e^t}{e^t+1}\right) = -\frac{d}{dt}\left(\frac{1}{e^{-t}+1}\right), \tag{4.6.32}$$

we have

$$\int_{-\infty}^\infty t^k \frac{d}{dt}\left(\frac{1}{e^t+1}\right) dt = \begin{cases} 0 & k \text{ is odd} \\ 2(1-2^{1-k})\Gamma(k+1)\zeta(k) & k\text{is even} \end{cases}. \tag{4.6.33}$$

From this follows the Sommerfeld expansion,

$$f_m^-(z) = (\log z)^m \sum_{k=0}^\infty \frac{2(1-2^{1-2k})\zeta(2k)}{\Gamma(m-2k+1)} (\log z)^{-2k}. \tag{4.6.34}$$

Applying this to the pressure and number density, we have

$$\beta p = \frac{g}{\lambda^3} f_{5/2}^-(z) = \frac{(\beta\mu)^{5/2}}{\Gamma(7/2)}\left(1 + \frac{5\pi^2}{8}(\beta\mu)^{-2} + \dots\right), \tag{4.6.35}$$

$$n = \frac{g}{\lambda^3} f_{3/2}^-(z) = \frac{(\beta\mu)^{3/2}}{\Gamma(5/2)}\left(1 + \frac{\pi^2}{8}(\beta\mu)^{-2} + \dots\right). \tag{4.6.36}$$

We know that at $T = 0$, $\mu = E_F$. Comparing this with the expression for the density, we find

$$E_F = k_B T \left(\Gamma(5/2) \frac{n\lambda^3}{g} \right)^{2/3}. \tag{4.6.37}$$

Using this, we can get the lowest order correction to the chemical potential:

$$\mu = E_F \left(1 - \frac{\pi^2}{12} (\beta E_F)^{-2} + \dots \right). \tag{4.6.38}$$

Note in particular that $\left(\frac{\partial \mu}{\partial T}\right)_{T=0} = 0$. Substituting this expansion into the equations for the pressure, we are led to

$$\beta p = \beta p_F \left(1 + \frac{5\pi^2}{12} (\beta E_F)^{-2} + \dots \right), \tag{4.6.39}$$

where $p_F = \frac{2}{5} n E_F$ is the pressure at zero temperature.

Problem 4.29 (J04T2)
Consider a gas of N nonrelativistic fermions with spin 1/2 and mass m initially at zero temperature and confined in a volume V_0.

a) Express the kinetic energy of the gas in terms of N and V_0.

b) What is the pressure of the gas? You can assume here that the gas is ideal.

c) Now the gas is allowed to expand to the volume $V_1 \gg V_0$ without any energy exchange with the outside world. Calculate the temperature of the gas after it will reach equilibrium due to weak interactions between the fermions.

d) What is the pressure of the gas in the final state?

Since the gas is at zero temperature, the fermions will fill the lowest N available states. Setting $V_0 = L^3$, the allowed values of k_x, etc. are $\frac{n\pi}{L}$, so the density of states is $\frac{2V_0}{(2\pi)^3}$ (the factor of 2 for spin degeneracy). This means the volume of momentum space occupied is $\frac{4\pi^3 N}{V_0}$; the volume will be a sphere, and so its radius is

$$k_F = \left(\frac{3\pi^2 N}{V_0} \right)^{1/3}. \tag{4.6.40}$$

The kinetic energy is

$$E = \int_0^{k_F} (4\pi k^2 \, dk) \frac{2V_0}{\pi^3} \frac{\hbar^2 k^2}{2m} = \frac{8\pi V_0}{5\pi^3} k_F^3 \frac{\hbar^2 k_F^2}{2m} = \frac{3}{5} N E_F, \tag{4.6.41}$$

where $E_F = \frac{\hbar^2 k_F^2}{2m}$ is the Fermi energy.

To find the pressure, we use the grand partition function,

$$\mathcal{G} = -k_B T \frac{2V}{\pi^3} \int d^3\boldsymbol{k} \, \log\left(1 + e^{\beta(\mu - \hbar^2 k^2/2m)}\right). \tag{4.6.42}$$

The pressure is given by

$$p = -\frac{\partial \mathcal{G}}{\partial V} = \frac{2k_B T}{\pi^3} \int d^3\boldsymbol{k} \, \log\left(1 + e^{\beta(\mu - \hbar^2 k^2/2m)}\right). \tag{4.6.43}$$

At zero temperature, $\mu = E_F$ and we can ignore the 1 in the logarithm, so this gives

$$p = \frac{\hbar^2}{m\pi^3} \int_0^{k_F} 4\pi k^2 \, dk \, \left(k_F^2 - k^2\right) = \frac{8\hbar^2 k_F^5}{15m\pi^2}. \tag{4.6.44}$$

The energy of the gas is

$$E = \frac{3}{2}pV = \frac{3}{2}p_F V \left(1 + \frac{5\pi^2}{12}\frac{k_B^2 T^2}{E_F^2} + \ldots\right). \tag{4.6.45}$$

Since $E_F \sim V^{-2/3}$, $p_F \sim V^{-5/3}$, and so maintaining constant energy implies

$$1 + \frac{5\pi^2}{12}\frac{k_B^2 T_f^2}{(E_F^{(0)})^2}\left(\frac{V_1}{V_0}\right)^{2/3} = \left(\frac{V_1}{V_0}\right)^{5/3}, \tag{4.6.46}$$

where $E_F^{(0)}$ is the Fermi energy at volume V_0. Solving, we find

$$T_f = \frac{E_F^{(0)}}{k_B}\left(\frac{V_1}{V_0} - \left(\frac{V_1}{V_0}\right)^{-2/3}\right)^{1/2} \approx \frac{E_F^{(0)}}{k_B}\sqrt{\frac{V_1}{V_0}}. \tag{4.6.47}$$

The final pressure is trivial to compute:

$$p_f = \frac{8\hbar^2 k_F^5}{15m\pi^2}\frac{V_0}{V_1}. \tag{4.6.48}$$

Problem 4.30 (M03T2)
Let us model a white dwarf star as a degenerate Fermi gas of electrons, supported against gravitational collapse by the electron degeneracy pressure. For simplicity, we will assume that the star is a sphere of radius R and uniform mass density containing N electrons, N protons, and N neutrons for an approximate total mass of $M = 2Nm_p$.

a) First, assume that the electrons are not relativistic. Find their Fermi energy and show that at absolute zero, their total kinetic energy is

$$U_k = \frac{3N(\hbar\pi)^2}{10m_e}\left(\frac{3N}{\pi V}\right)^{2/3} \tag{4.6.49}$$

where V is the volume of the star. (Note that the total kinetic energy of the nucleons is much smaller than that of the electrons.)

b) The gravitational binding energy of a uniform density sphere is

$$U_{\text{grav}} = -\frac{3GM^2}{5R}. \tag{4.6.50}$$

Find the equilibrium radius for the white dwarf. Eliminate N. How does this radius depend on the mass?

c) If instead the electrons are highly relativistic, so that their energy and momentum are related by $\epsilon = cp$, then find the Fermi energy and show that the total kinetic energy is now

$$U_k = \frac{3N\hbar\pi c}{4}\left(\frac{3N}{\pi V}\right)^{1/3} \tag{4.6.51}$$

d) Under what conditions is a highly relativistic degenerate electron star unstable against collapse? This is called the Chandrasehkar limit. A star that violates the limit will collapse into a neutron star or black hole, depending on whether neutron degeneracy pressure can hold up the star.

We derived the kinetic energy of a degenerate Fermi gas in the last problem, no use repeating it in great detail:

$$U_k = \frac{3}{5}E_F = \frac{3\hbar^2}{10m_e}k_F^2 = \frac{3\hbar^2}{10m_e}\left(\frac{3\pi^2 N}{V}\right)^{2/3}. \tag{4.6.52}$$

Substituting $V = \frac{4\pi}{3}R^3$, the total energy is

$$E = \frac{3\hbar^2}{10m_e R^2}\left(\frac{9\pi N}{4}\right)^{2/3} - \frac{3G(2Nm_p)^2}{5R}. \tag{4.6.53}$$

Minimizing with respect to R, we find

$$R_{\text{eq}} = \frac{3\pi}{8}\left(\frac{3\pi^2}{2N^4}\right)^{1/3}\frac{\hbar^2}{8Gm_e m_p^2}. \tag{4.6.54}$$

If the electrons are relativistic, then we have

$$U_k = \int_0^{k_F}(4\pi k^2\,dk)\frac{2V}{(2\pi)^3}\hbar ck = \frac{\hbar cV}{4\pi^2}k_F^4 = \frac{3N\hbar c}{4}\left(\frac{3\pi^2 N}{V}\right)^{1/3}. \tag{4.6.55}$$

With the relativistic expression, the kinetic energy also scales as R^{-1}, so we run into trouble when the overall coefficient is negative:

$$\frac{3(M/2m_p)\hbar c}{4}\left(\frac{9\pi(M/2m_p)}{4}\right)^{1/3} < \frac{3GM^2}{5}, \tag{4.6.56}$$

or more succinctly,

$$M > \frac{15\sqrt{5\pi}}{64m_p^2}\left(\frac{\hbar c}{g}\right)^{3/2}. \tag{4.6.57}$$

4.7 Additional Problems

Problem 4.31 (J05T3)
A thermodynamic system has the following relation between its entropy S, volume V, internal energy U, and particle number N:

$$S(U,V,N) = \gamma(UVN)^{1/3}, \tag{4.7.1}$$

where γ is a constant.

a) Derive a relation connecting U, N, V, and the temperature T.

b) Find the heat capacity C_{VN} at constant V and N, as a function of V, N, and temperature T.

Now assume you are given two identical bodies with the above properties. N and V are the same for both, and are fixed, but the two bodies have different initial temperatures, T_1 and T_2.

c) If the two bodies are placed in thermal contact, and left alone until heat flow ceases and equilibrium is reached, what is their common final temperature T_f?

d) If the flow of heat between the bodies is used to drive an engine that does the maximum possible amount of useful work $W_{\max}$ before the two bodies reach a common final temperature T_f', what is that temperature? What is $W_{\max}$?

We have $U = \frac{S^3}{\gamma^3 VN}$, and so

$$T = \left(\frac{\partial U}{\partial S}\right)_{V,N} = \frac{3S^2}{\gamma^3 VN}. \tag{4.7.2}$$

This relation implies $S = \sqrt{\frac{\gamma^3 VNT}{3}}$, so the heat capacity is

$$C_{VN} = T\left(\frac{\partial S}{\partial T}\right)_{V,N} = \frac{1}{2}\sqrt{\frac{\gamma^3 VNT}{3}}. \tag{4.7.3}$$

Let $\alpha = \frac{1}{2}\sqrt{\frac{\gamma^3 VN}{3}}$, so $C_{VN} = \alpha T^{1/2}$. Then when the bodies are placed in thermal contact, we have

$$dT_1 = -\frac{đQ}{\alpha T_1^{1/2}}, \qquad dT_2 = \frac{đQ}{\alpha T_2^{1/2}}. \tag{4.7.4}$$

Integrating, we have

$$T_f = \left(\frac{T_1^{3/2} + T_2^{3/2}}{2}\right)^{2/3}. \tag{4.7.5}$$

If we use a Carnot engine to equilibrate, then the heat transfer at each source is proportional to the temperature at each source, so we have

$$dT_1 = -\frac{T_1^{1/2}\,dx}{\alpha}, \qquad dT_2 = \frac{T_2^{1/2}\,dx}{\alpha}. \tag{4.7.6}$$

By a similar integration we find

$$T_f' = \left(\frac{T_1^{1/2} + T_2^{1/2}}{2}\right)^2. \tag{4.7.7}$$

We can determine the work done by balancing the total energy:

$$\begin{aligned} W_{\max} &= \int_0^{T_1} \left(\alpha T^{1/2}\right) dT + \int_0^{T_2} \left(\alpha T^{1/2}\right) dT - \int_0^{T_f'} \left(2\alpha T^{1/2}\right) dT \\ &= \frac{2}{3}\alpha \left(T_1^{3/2} + T_2^{3/2} - 2\left(\frac{T_1^{1/2} + T_2^{1/2}}{2}\right)^3\right). \end{aligned} \tag{4.7.8}$$

Problem 4.32 (M98T3)
Using the anharmonic potential $V(x) = cx^2 + gx^3 + fx^4$, for a one-dimensional classical harmonic oscillator, find the approximate heat capacity at low temperatures including terms of order T.

The states of the system are pairs (x, p), and so the partition function is

$$Z = \int dp\, e^{-\beta \frac{p^2}{2m}} \int dx\, e^{-\beta(cx^2+gx^3+fx^4)} = \sqrt{2\pi m k_B T} \int dx\, e^{-\beta(cx^2+gx^3+fx^4)}. \tag{4.7.9}$$

It's unclear which term is dominant in the potential, but this looks hard unless $g, f \ll c$ (also what is up with the labeling of the coefficients? not even alphabetical…), so we'll make that assumption. Then we have

$$\int dx\, e^{-\beta(cx^2+gx^3+fx^4)} \approx \int dx\, e^{-\beta cx^2} \left(1 - \beta(gx^3 + fx^4)\right) = \sqrt{\frac{\pi}{\beta c}} \left(1 - \frac{3f}{4\beta c^2}\right). \tag{4.7.10}$$

In total, this gives

$$Z = \frac{\pi}{\beta}\sqrt{\frac{2m}{c}} \left(1 - \frac{3f}{4\beta^2 c^2}\right). \tag{4.7.11}$$

Thus, the heat capacity is

$$C = -T\frac{\partial^2 F}{\partial^2 T} = k_B - \frac{3fk_B^2}{2c^2}T + \mathcal{O}\left(T^2\right). \tag{4.7.12}$$

Problem 4.33 (J07T3)
Consider spin waves in an isotropic ferromagnetically ordered crystal. These are waves in which the spins on each atom oscillate in space and time. Just as with sound waves, the spin waves can be quantized and they can store internal energy in a crystal lattice. However, these waves have a different relation between frequency and wavenumber than do sound waves. In particular, at low wavenumber,

$$\omega(k) = Ak^2, \tag{4.7.13}$$

where A is a constant. Consider a crystal containing N spins in thermal equilibrium at temperature T.

a) What is the average energy in a spin wave mode of frequency ω? (Neglect the zero-point energy of the mode).

b) At low temperatures, the heat capacity of the spin wave system in the crystal is proportional to T^α. What is the numerical value of α?

c) If the material is a metal, do the spin waves give the dominant contribution to the heat capacity in the low temperature limit? What if the material is an insulator? Explain both of your answers.

The average occupancy of a spin mode of frequency ω is

$$\langle n(\omega)\rangle = \frac{1}{e^{\beta\hbar\omega} - 1}, \tag{4.7.14}$$

so the average energy is

$$\langle E(\omega)\rangle = \frac{\hbar\omega}{e^{\beta\hbar\omega} - 1}. \tag{4.7.15}$$

At low temperatures, we have $\langle E(\omega)\rangle = \hbar\omega e^{-\beta\hbar\omega}$. Thus, the energy due to spin waves is

$$E \propto \int d^3\boldsymbol{k} \left(\hbar A k^2\right) e^{-\beta\hbar A k^2} \propto \beta^{-5/2}. \tag{4.7.16}$$

Thus, $C = \frac{\partial E}{\partial T} \propto T^{3/2}$.

In a metal, the Fermi electron gas gives a contribution $C \propto T$ at low temperatures, so the spin waves are subdominant. In an insulator, the other contribution is the phonons which give $C \propto T^3$ at low temperatures, so the spin waves are dominant.

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