Évariste Galois and the Solvability of Equations

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Outline

Math on a Clock

- Modular arithmetic is arithmetic with an upper bound
- Example, modulo 5:

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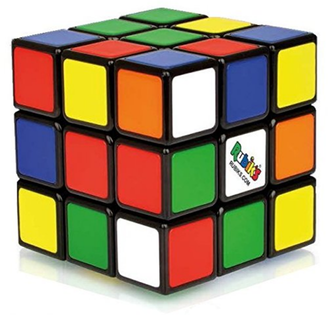
- Basically: a set ($\{0, 1, 2, 3, 4\}$), and an operation ($+$)

Rubik's Cube

- There are a set of operations you can perform on a Rubik's cube
- Performing two in sequence is equivalent to another operation



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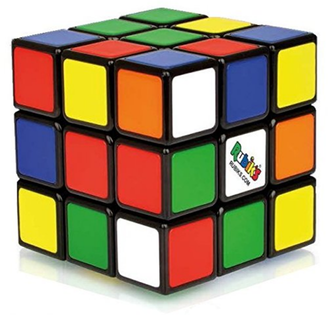

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- A group G consists of:
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- More examples:
  - Permutations of students
  - Symmetries of a pentagon
  - Rotations of a sphere

Groups as Symmetries

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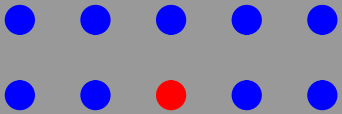
- *All groups can be constructed in this way*
- Group theory is a powerful method for studying symmetries in a general way

Group Actions

- Even if a group comes from symmetries, the group itself is not tied to an object
- A *group action* specifies how elements of a group act on some object
- Example: group of 3D rotations has an action on 3D space
- If part of the object is left alone by an element of the group, we say it is a *fixed point* of that element

Subgroups

- Groups can have smaller groups, called subgroups living inside them
- When a symmetry is partially broken, the leftover symmetry group is a subgroup of the original
- Example: permutations



Conjugacy Classes

- A group element g is said to be conjugate to h if $g = xhx^{-1}$ for some $x \in G$
- Elements of a group split into conjugacy classes of conjugate elements
- Example: rotations of a sphere are conjugate if they are by the same angle

$$R_z(45^\circ) = R_y(-90^\circ)R_x(45^\circ)R_y(90^\circ)$$

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- The rotations $SO(3)$ have no nontrivial normal subgroup, so it is called *simple*

Finite Simple Group of Order Two

https://www.youtube.com/v/UTby_e4-Rhg?rel=0

Fields

- Groups are powerful, but can't capture everything
- Ordinary math involves two operations, multiplication and addition
- This can be captured by *fields*. A field is:
  - An Abelian group for the addition operation
  - A mutliplication operation, invertible for everything except 0, and distributive over addition
- Examples: $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$

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- This is a *field extension* over the rationals, denoted $\mathbb{Q}(\sqrt{2})$

Degree of Extension

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- Degree is written as $[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 2$

Construction by Compass and Straightedge

- You may have heard that it is impossible to trisect an angle via compass and straightedge
- To prove this, you can show that a compass and straightedge only permits degree 2 field extensions
- Trisection of angles requires a degree 3 field extension

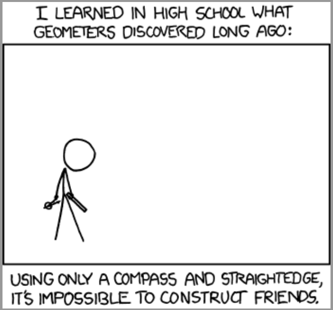

Figure: Impossiblity theorems are sad.

Splitting Fields

- Recipe for a field extension: take a polynomial, and adjoin all its roots to the base field
- Example: $x^2 - 2$ over $\mathbb{Q}$. Adjoin $\pm\sqrt{2}$ to $\mathbb{Q}$ to form $\mathbb{Q}(\sqrt{2})$
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- In the splitting field, a polynomial splits: $x^2 - 2 = (x + \sqrt{2})(x - \sqrt{2})$
- The degree $[\mathbb{Q}(\alpha) : \mathbb{Q}]$ equals the degree of the *minimum polynomial* of α over $\mathbb{Q}$

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- Total degree:
$[\mathbb{Q}(\sqrt[4]{2}, i) : \mathbb{Q}] = [\mathbb{Q}(\sqrt[4]{2}, i) : \mathbb{Q}(\sqrt[4]{2})][\mathbb{Q}(\sqrt[4]{2}) : \mathbb{Q}] = 4 \cdot 2 = 8$

Field Automorphisms

- Consider the field extension $\mathbb{Q}(\sqrt{2})$ and the map $f(a + b\sqrt{2}) = a - b\sqrt{2}$
- This respects multiplication and addition:

$$f(x) + f(y) = f(x + y)$$
$$f(x) \cdot f(y) = f(x \cdot y)$$

- It also leaves elements of the base field $\mathbb{Q}$ alone
- We call such a map a $\mathbb{Q}$-automorphism

Field Automorphisms

- Remember the recipe for groups? Take an object, look at transformations leaving its structure invariant.
- Field automorphisms are exactly this kind of construction
- The F-automorphisms of an extension E over a field F are called the *Galois group* of E over F

Enter Galois

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Figure: Évariste Galois at age 15.

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- Arrested for threatening the life of King Louis Philippe; later served six months in prison, where he continued his mathematical work

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- But, beforehand, he had collected all his mathematical thoughts in a letter
- "This letter, if judged by the novelty and profundity of ideas it contains, is perhaps the most substantial piece of writing in the whole literature of mankind."

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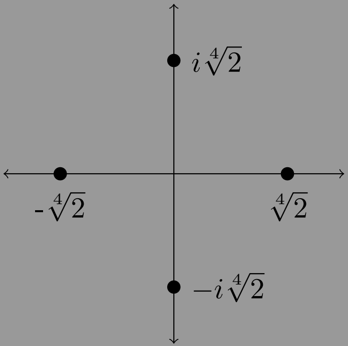
The Galois Correspondence

- We have seen the Galois group, $\mathrm{Gal}(E/F)$: the group of F-automorphisms of E
- Galois showed that the subgroups of $\mathrm{Gal}(E/F)$ correspond to field extensions living between E and F
- This is the fundamental theorem of Galois theory

Example: $x^4 - 5x^2 + 6$

- We have already seen that the splitting field of $x^4 - 2$ is $\mathbb{Q}(\sqrt[4]{2}, i)$
- What are the $\mathbb{Q}$-automorphisms?



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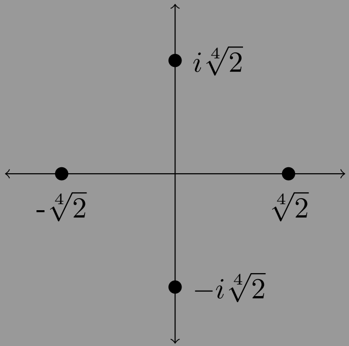

- The automorphisms are the symmetries of the square, generated by rotations $\sqrt[4]{2} \to i\sqrt[4]{2}$ and reflections $i \to -i$

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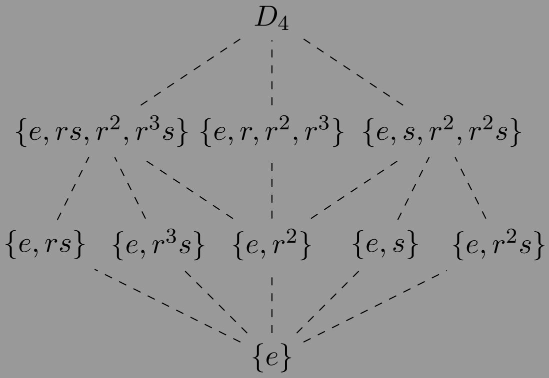

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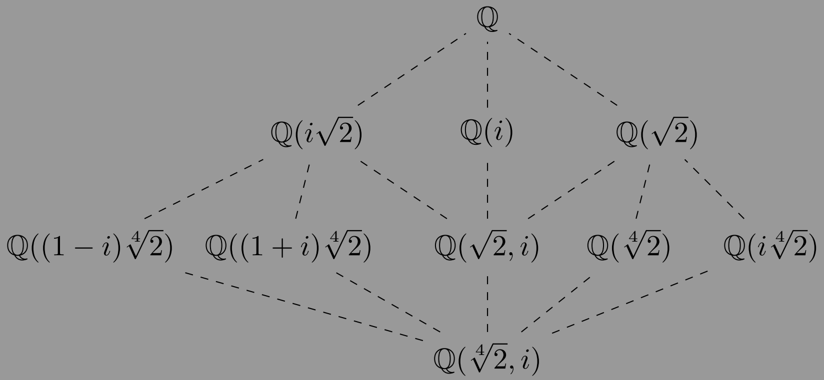
- What happens if we make this association for each subgroup?

Example: $x^4 - 2$

- Every field extension between $\mathbb{Q}$ and $\mathbb{Q}(\sqrt[4]{2}, i)$ appears
- Inclusion is reversed



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- Such a sequence does not exist for all groups
- Galois shows that the Galois group for the general quintic polynomial does not have such a sequence
- This proves the Abel-Ruffini theorem: there exists no general method of solution by radicals for quintic and higher polynomials