Richard Feynman, the Path Integral, and Least Action Principles

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Outline

1. Least Action Principles
2. Quantum Mechanics
3. Path Integral Formulation

Action in Mechanics

- Newton discovered that $\boldsymbol{F} = m\boldsymbol{a}$. This tells us how particles will move at each point in time
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- What governs the overall trajectories?
- Newton's laws are equivalent to minimizing a quantity, *action*, which depends on whole trajectories

Actions and Lagrangians

- No point in time is more special than the others, so the action should care equally about every time slice
- Write the action as an integral (sum) over all times:

$$S = \int \mathcal{L}\, dt$$

- $\mathcal{L}$ is the *Lagrangian*

Lagrangians and Theories

- Mathematics gives a recipe for converting Lagrangians to equations of motion:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{\partial \mathcal{L}}{\partial x}.$$

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- Example: in mechanics, $\mathcal{L} = T - V$, the difference of kinetic and potential energies
- Resulting equations of motion:
$$m\ddot{x} = -\frac{\partial V}{\partial x} = F$$
Newton's second law!

Electromagnetism

- Sometimes this is called "Lagrangian mechanics." This obscures a crucial fact: *all known physical theories can be approached in the same way*
- Example: Lagrangian for electromagnetism is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + 4\pi j_\mu A^\mu = \frac{1}{2}(E^2 - B^2)$$

- Using the recipe, this gives the dynamic Maxwell equations in a vacuum:

$$\boldsymbol{\nabla}\cdot\boldsymbol{E} = 4\pi\rho, \qquad \boldsymbol{\nabla}\times\boldsymbol{B} = \frac{4\pi}{c}\boldsymbol{j} + \frac{1}{c}\frac{\partial\boldsymbol{E}}{\partial t}$$

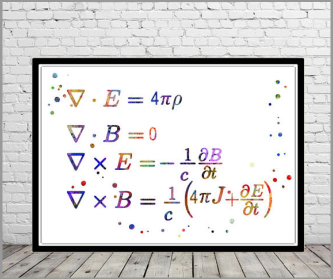

Figure: Maxwell's equations on Etsy

Why care?

- If we already know Newton's laws and Maxwell's laws, why generate them with a Lagrangian?
- Lagrangians can be extremely powerful for understanding conservation laws
- Look again at the equations of motion:

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- Independence of x generates a conserved quantity $\frac{\partial \mathcal{L}}{\partial \dot{x}}$
- This is *Noether's theorem*: a symmetry generates a conservation law

Conservation of Energy

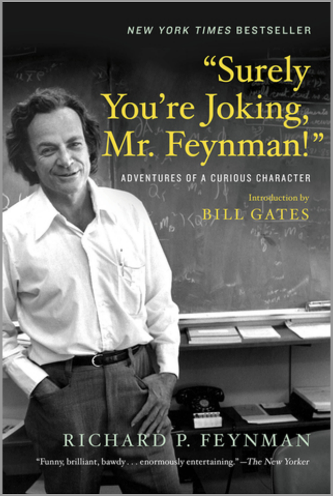
- By applying Noether's theorem to time translation symmetry, we obtain conservation of energy
- The energy is given by

$$\mathcal{H} = \sum \frac{\partial \mathcal{L}}{\partial \dot{x}_i} \dot{x}_i - \mathcal{L}$$

- This is also called the *Hamiltonian*

Break: Feynman and the Door

- "My masterpiece of mischief happened at the fraternity."



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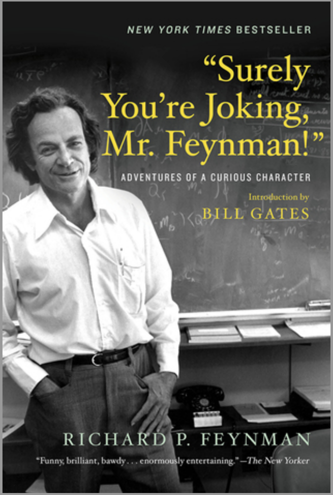
- "My masterpiece of mischief happened at the fraternity."
- "Feynman! Did you take the doors?" "Oh, yeah!", I said. "I took the door. You can see the scratches on my knuckles here, that I got when my hands scraped against the wall as I was carrying it down into the basement."



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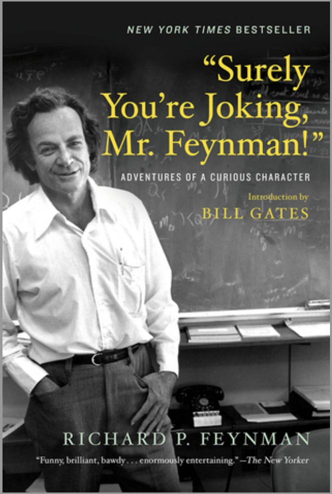

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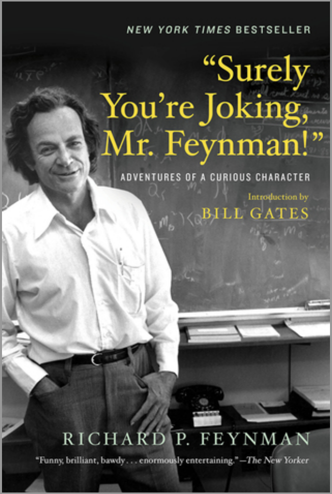
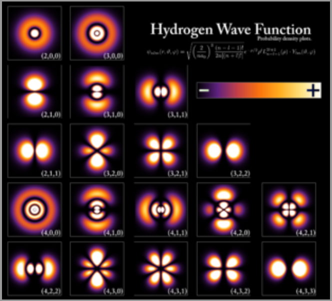
- "Feynman, did you take the door?" "Yeah, I took the door." "Cut it out, Feynman; this is serious!"

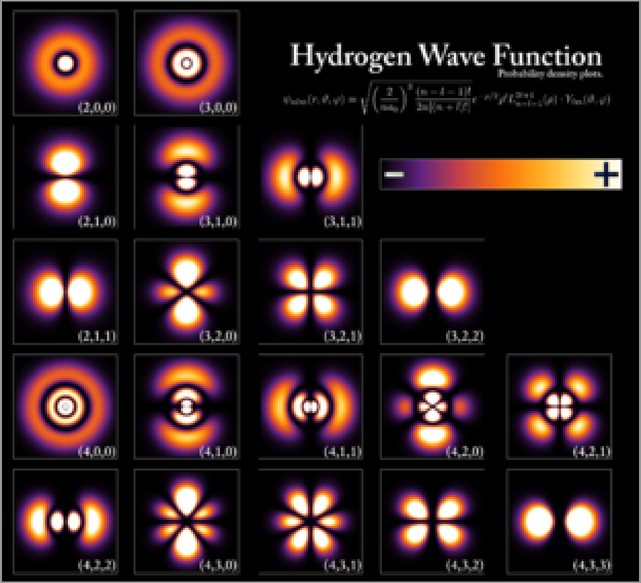


Quantum States

- Classical states have definite values for every observable
- Quantum theory allows for *superpositions* of definite states
- When expressed in terms of position states, quantum states are called *wavefunctions*



Example: Hydrogen



Born Interpretation

- Superpositions are interpreted as specifying probabilities
- Measurement of a quantum system forces it to collapse into a definite state, based on the probabilities
- Until it is measured, the system evolves as a superposition

Time Evolution

- Quantum systems evolve according to the Schrödinger equation:

$$i\hbar \frac{\partial}{\partial t} |\psi\rangle = \mathcal{H} |\psi\rangle$$

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- If a system is in a definite energy state, it evolves in a trivial way
- In the most general case, time evolution can be expressed in terms of an operator U:

$$\left|\psi(t_1)\right\rangle = U(t_1, t_0)\left|\psi(t_0)\right\rangle$$

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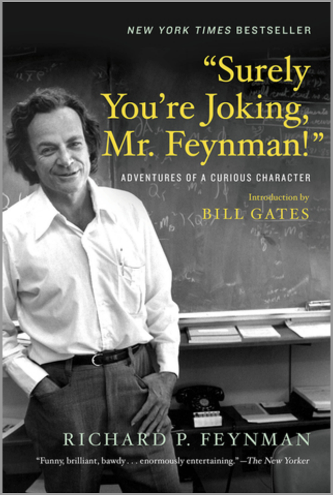
- Where's the connection to the classical Lagrangian? It is present, but not evident.
- "Beauty is the first test." This fails.

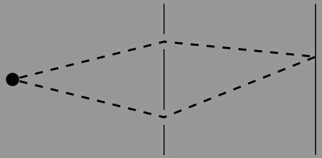
Break: Feynman and the Safe

- Feynman worked on the Manhattan project in Los Alomos during World War II
- "To demonstrate that the locks meant nothing, whenever I wanted somebody's report and they weren't around, I'd just go in their office, open the filing cabinet, and take it out. When I was finished I would give it back to the guy: "Thanks for your report."'
- "I opened the safes which contained all the secrets to the atomic bomb: the schedules for the production of the plutonium, the purification procedures, how much material is needed, how the bomb works..."



Double Slit Experiment

- Since quantum mechanics allows for superpositions, an electron can go through two slits *at once* to get to a screen on the opposite side



Breaking up Time Evolution

- Before we wrote time evolution in a single step:

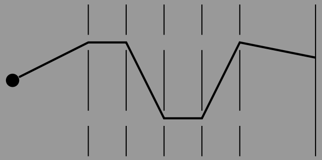
$$|\psi(t_1)\rangle = U(t_1, t_0)\,|\psi(t_0)\rangle$$

- We could also break up the time evolution into N steps:

$$|\psi(t_N)\rangle = U(t_N, t_{N-1})U(t_{N-1}, t_{N-2})\cdots U(t_1, t_0)\,|\psi(t_0)\rangle$$

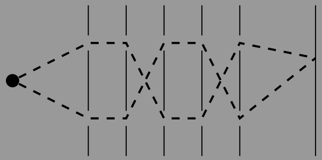
Slicing up the Double Slit

- Breaking up time evolution is somewhat like making a measurement at multiple points along a path
- One way of making a measurement is to use a double slit apparatus and check which slit the particle goes through



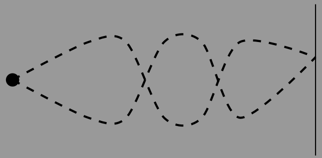
Turning off Measurements

- If we don't make measurements at the slits, the particle can be in a superposition
- The particle can take all possible paths through the slits



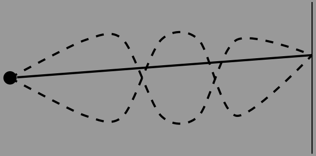
Fine-Grained Slits

- Instead of a double slit, how about three? Four? N?
- By adding slits, we increase the freedom of the particle to take multiple paths
- In the limit, there are no slits, just an open space



Summing over Paths

- We now have a new picture of quantum mechanics: particles taking all possible paths from point A to point B
- If this is how particles work, how come classically we see a single path?

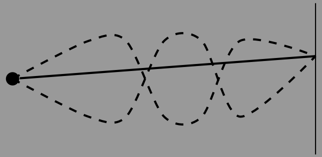


Classical Limit

- In the sum over paths, each one is weighted by a *phase factor*. Formally,

$$\langle f|i\rangle = \int \mathcal{D}x(t)e^{iS[x(t)]}$$

- Since the quantity S varies, these factors cancel out



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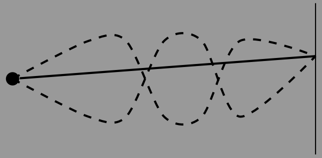
- *Except* at the path where S is minimized



Classical Limit

- The quantity S is the *classical action for the path*
- The path integral reproduces the principle of least action