Leonhard Euler: Master of us All

Ross Dempsey

Department of Physics and Astronomy
Johns Hopkins University

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Outline

Euler the Great

- Euler was possibly the most productive mathematician ever
  - Created enough work to fill 74 volumes
  - Over 866 items, including cutting-edge papers, expository books, technical manuals
  - Many more volumes of correspondence
- Contributed to nearly every area of mathematics and physics in his time, and spawned several new areas

Figure: Leonhard Euler, 1707–1783

Peter the Great

- Peter the Great was intent on modernizing Russia, which included creating an Academy of Sciences
- Russia had no native talent, so foreign scientific minds were imported, including the young Euler in 1727
- In 1733, became head mathematician and married Katharina Gsell
- Became blind in one eye in 1738; “Now I will have fewer distractions”

Figure: Peter the Great, 1672–1725

Frederick the Great

- Frederick the Great aspired to be a "philosopher-king" of Prussia, and revived the Berlin Academy of Sciences
- Political turmoil was causing problems in St. Petersburg, so Euler moved to Berlin and joined the Academy
- Russians maintained great respect for Euler
- Frederick the Great appreciated Euler's accomplishments but wasn't fond of him personally

Figure: Frederick the Great, 1712–1786

Catherine the Great

- Catherine the Great became empress of Russia in a coup d'etat of her brother
- Sought to revive the Academy of Sciences, which amounted to bringing Euler back
- Euler returned, welcomed as a celebrity
- His mathematical output continued until the day of his death, despite full blindness

Figure: Catherine the Great, 1729–1796

Basel Problem

- Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \dots.$$

- Finding the exact sum was posed by Pietro Mengoli in 1644
- Euler solved the problem in 1734, when he was 28

Taylor Approximation

- Every function can be approximated by polynomials
- The *Taylor series* of a function is an infinite polynomial
- In some cases, a function is equal to its Taylor series. This is true of the sine function:

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \dots$$

Infinite Products

- Finite polynomials can be factored into linear polynomials, one for each root:

$$x^3 - 4x^2 - 11x + 30 = (x-2)(x+3)(x-5)$$

- This can also be done for infinite polynomials. What are the roots of

$$x - \frac{x^3}{6} + \frac{x^5}{120} - \ldots ?$$

Infinite Products

- Since $\sin x = 0$ at $x = \pi n$, these are the roots. So the factorization looks like

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \ldots = x(x - 2\pi)(x + 2\pi)(x - 4\pi)(x + 4\pi)\cdots$$

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- As written, this would give an infinite linear term. So, rewrite as

$$\sin x = x\left(1 - \frac{x}{\pi}\right)\left(1 + \frac{x}{\pi}\right)\left(1 - \frac{x}{2\pi}\right)\left(1 + \frac{x}{2\pi}\right)\cdots$$

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- Now the linear term is fixed to x. The quadratic term is zero. What is the cubic term?

Infinite Products

- Write each factor $\left(1 - \frac{x}{\pi n}\right)\left(1 + \frac{x}{\pi n}\right)$ as $1 - \frac{x^2}{\pi^2 n^2}$:

$$\sin x = x\left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{\pi^2 \cdot 2^2}\right)\left(1 - \frac{x^2}{\pi^2 \cdot 3^2}\right)\cdots$$

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- The cubic term is

$$-\frac{1}{2\pi}-\frac{1}{2\pi\cdot 2^2}-\cdots=-\frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}$$

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- By comparing with the Taylor series, Euler concluded

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}.$$

Bridges of Königsberg

- The city of Königsberg had seven bridges over the river Pregel
- Euler wondered if one could walk through the city, crossing each bridge exactly once
- Do you see a way to do this?

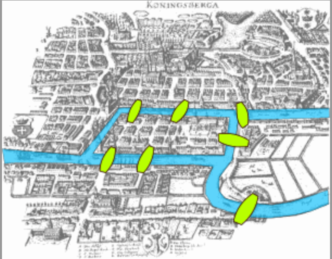


Figure: The bridgs of Königsberg in 1736.

Graph Theory

- Euler invented graph theory to solve this problem. Graph theory is now an important part of mathematics
- A *graph* is a collection V of vertices, and a collection E of edges between them
- What are the vertices and edges in Königsberg?

Königsberg Graph

- Islands are vertices, bridges are edges
- Representing the city as a graph allows us to ignore all the particulars of Königsberg and focus on the underlying mathematical question
- Is there a path on this graph which crosses every edge exactly once?

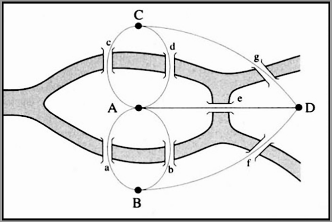
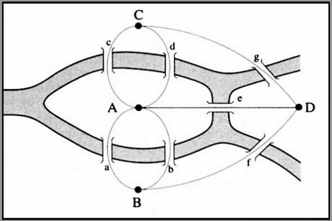


Figure: The bridgs of Königsberg form a graph.

Eulerian Path

- Such a path is today called an Eulerian path
- A simple criterion exists for determining if a graph has an Eulerian path



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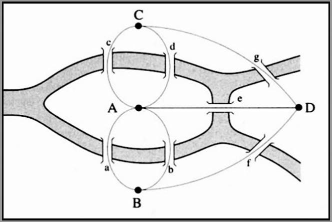

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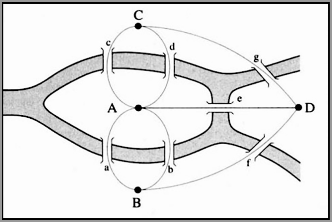
- How do vertices with odd degree affect the formation of an Eulerian path?



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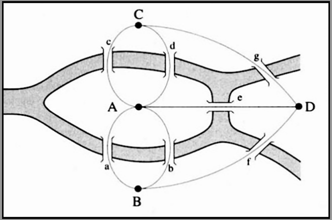
Eulerian Circuit

- Euler realized that crossing each bridge exactly once in a closed loop requires an even degree for each vertex
- Likewise, to go from one vertex to another on an Eulerian path, the start and finish vertices should have odd degree and the rest should have even degree
- This is considered the first theorem of graph theory



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Modern Königsberg

- The city of Königsberg is now called Kaliningrad
- During World War II, the city was bombed, and two bridges did not survive
- One has apparently been added
- Is there an Eulerian path now? Is there an Eulerian circuit?

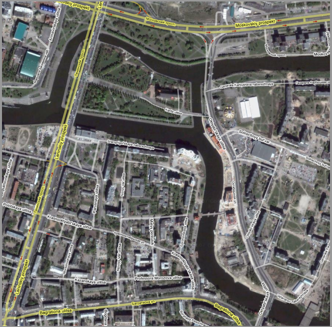

Figure: Modern-day Königsberg bridges

Brussels Sprouts

- The Eulerian circuit problem is pretty easy
- Graph theory can be much, much more subtle
- An example of medium difficulty: Brussels Sprouts

Polyhedra

- A polyhedron is the 3D equivalent of a polygon
- Name some polyhedra

Figure: Some wooden polyhedra

Platonic Solids

- If all the sides of a polyhedron are identical, it is a Platonic solid
- The five Platonic solids, with their vertex, edge, and face counts, are listed below. Do you notice anything?

	V	E	F
Tetrahedron	4	6	4
Cube	8	12	6
Octahedron	6	12	8
Dodecahedron	20	30	12
Icosahedron	12	30	20

Euler's Formula

- It's okay if you didn't – mathematicians missed this for millenia
- Here they are again, with another quantity listed

	V	E	F	$V - E + F$
Tetrahedron	4	6	4	2
Cube	8	12	6	2
Octahedron	6	12	8	2
Dodecahedron	20	30	12	2
Icosahedron	12	30	20	2

Topology

- Why does this work for all polyhedra?
- In modern mathematics, we express this fact as $\chi(S^2) = 2$
  - S^2 denotes the 2-sphere (the sphere which lives in 3-space)
  - χ is the *Euler characteristic*, which in two dimensions is $V - E + F$
- The Euler characteristic χ is a *topological invariant*

Topology

- Topology does not care about the exact structure of an object
- Manifolds, like S^2, can be stretched and deformed in continuous ways
- All polyhedra are topologically equivalent to one another and to S^2

Figure: In topology, coffee cups are equivalent to donuts

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  - $2E = 5n_5 + 6n_6$
  - $F = n_5 + n_6$
- Euler tells us

$$V - E + F = \frac{n_5}{6} = 2,$$

so $n_5 = 12$