Albert Einstein, Black Holes, and Gravitational Waves

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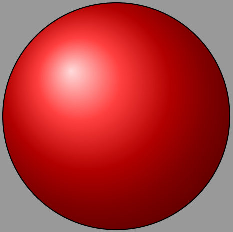
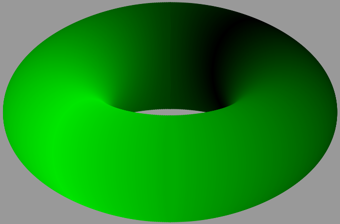
Outline

What is curvature?

- Which of the following spaces are curved?





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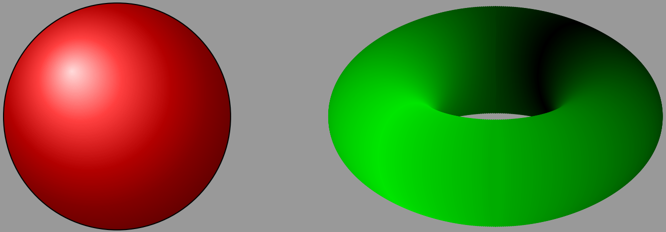

- The sphere is curved, but the torus is not.

Curvature 1: Triangles

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  - Answer: 270°

Curvature 1: Triangles

- In the 1820s, Gauss attempted to measure the curvature of space itself by forming a great triangle between three mountains
- Spoiler: space itself *is* curved. Why did Gauss's measurement not indicate this?

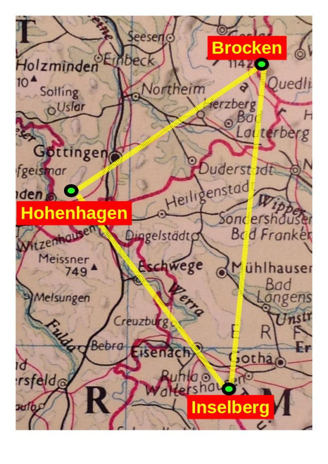

Figure: Gauss measured the angles of a triangle between three mountains.

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- If parallel transport can change the direction of a vector, space is curved.

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- Curvature is defined at each point. The angle deficit or the parallel transport around a path is related to the curvature contained within that path.
- For a sphere, curvature is constant, so these effects are related to area. What's the area of the spherical triangle?

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- Morally: the quantity of interest is the scalar curvature K.

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$$ds^2 = dx^2 + dy^2 + dz^2.$$

- This formula might look different depending on the coordinates. So, generalize:

$$ds^2 = \sum_{i=1}^{3} \sum_{j=1}^{3} g_{ij}\, dx^i\, dx^j.$$

Metrics

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- Moral: metrics themselves don't tell us about the space

Curvature in a Metric

- Since the same flat space can have many different metrics, it's not easy to detect when a space itself is curved.
- What if the metric is constant?

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- A constant metric is always flat space in disguise. In this case, use coordinates $u = \frac{\sqrt{3}}{2}(dx + dy)$, $v = \frac{1}{2}(dx - dy)$. Then

$$\begin{aligned} du^2 + dv^2 &= \frac{3}{4}(dx^2 + 2dx\,dy + dy^2) + \frac{1}{4}(dx^2 - 2dx\,dy + dy^2) \\ &= dx^2 + dx\,dy + dy^2. \end{aligned}$$

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- Moral: curvature must come from changes in the metric

Christoffel Symbols

- The *Christoffel symbols* are related to the rate of change of a metric:

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl} \left(\partial_l g_{ij} - \partial_i g_{lj} - \partial_j g_{il} \right)$$

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- Downside: Christoffel symbols depend on coordinate system, like the metric

Riemann Tensor

- Christoffel symbols can be combined to form an object which tells us about curvature, regardless of the underlying coordinate system
- This is the *Riemann curvature tensor*:

$$R^i_{jkl} = \partial_k \Gamma^i_{lj} - \partial_l \Gamma^i_{kj} + \Gamma^i_{ka}\Gamma^a_{lj} - \Gamma^i_{la}\Gamma^a_{kj}$$

- If any component of the Riemann tensor is nonzero, the space is curved

Ricci Scalar

- To go from a matrix to a scalar, you can sum its diagonal entries, obtaining the *trace*
- The trace is independent of the basis for the matrix
- Likewise, with the Riemann tensor, we can form a single number which is independent of the coordinate system:

$$R = R^i_i, \qquad R_{ij} = R^a_{iaj}.$$

- This number R is called the *Ricci scalar*

Summary

- Metric g_{ij}: tells us how to measure distance near a point
- Christoffel symbol Γ^i_{jk}: tells us how the metric changes
  - Also how to transport vectors, and how to find the shortest path between points
- Ricci tensor R^i_{jkl}: complete, coordinate-independent information about curvature
- Ricci scalar R: a single number indicating curvature

Einstein-Hilbert Action

- Theories of physics can be specified by a single quantity, the *action*
- The action should be sampled uniformly over space and time, so we write it as the integral of a *Lagrangian*
- The Lagrangian is a scalar and must respect the symmetry of the theory
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- Einstein tells us (i) spacetime is curved and (ii) coordinate systems don't matter (relativity)
- Natural conclusion: the Lagrangian is the Ricci scalar R of spacetime

Einstein Field Equations

- Using the Lagrangian, it is possible to construct the equation of motion:
$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0.$$
- The right hand side is zero only in a vacuum; in the presence of matter, it is related to the *stress-energy tensor* $T_{\mu\nu}$:
$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}.$$
- The constant $8\pi G$ is chosen to ensure agreement with Newton's law of gravity

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- Einstein tells us:

$$a^i = \Gamma^i_{jk} v^j v^k \sim -\frac{1 + g_{tt}}{2}$$

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- Interpret $-\frac{1}{2}(1 + g_{tt})$ as ϕ, then the theories agree in the weak gravity limit

Meaning of Einstein's Equations

- Consider the source-free Einstein equation, $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0$
- $R_{\mu\nu}$ are derivatives of $\Gamma^{\sigma}_{\mu\nu}$, which are derivatives of $g_{\mu\nu}$
- Einstein's equations are complicated, nonlinear partial differential equations of the metric
- One solution is flat space, $R_{\mu\nu} = 0$ and $R = 0$
- Differential equations can have many solutions, but finding them is extremely difficult in this case

Schwarzschild Solution

- Shortly after Einstein published his equation, Schwarzschild found a static, spherically symmetric solution:

$$ds^2 = -\left(1 - \frac{r_s}{r}\right) dt^2 + \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\phi^2.$$

- The radius r_s is called the *Schwarzschild radius*

$$\phi = -\frac{1}{2}(1 + g_{tt}) = -\frac{r_s}{2r} \implies r_s = 2GM$$

- Note that ds^2 becomes undefined at $r = r_s$. What does this mean?

Black Holes

- The surface $r = r_s$ is an *event horizon*
- What happens to an observer who crosses the event horizon?

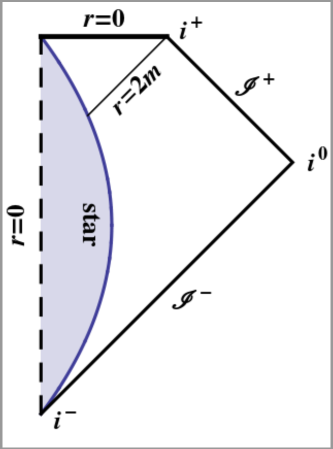

Figure: A Penrose diagram for a star collapsing into a black hole.

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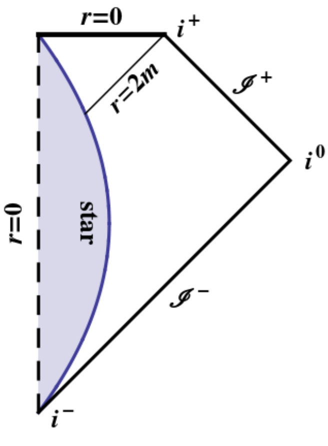

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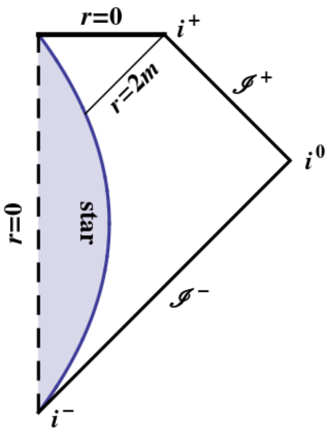
- The observer will reach $r = 0$ within a finite time, after which *spacetime ends*



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Other Black Holes

- Black holes are fully specified by mass, charge, and angular momentum
- Fun fact: existence of charged black holes implies gravity is the weakest force
- Rotating black holes are called Kerr black holes. Kerr metric is very complicated

Kerr Metric

- Black holes are fully specified by mass, charge, and angular momentum
- Rotating black holes are called Kerr black holes. Kerr metric is extremely complicated
- The black hole has inner and outer event horizons
- An *ergosphere* exists outside the event horizons

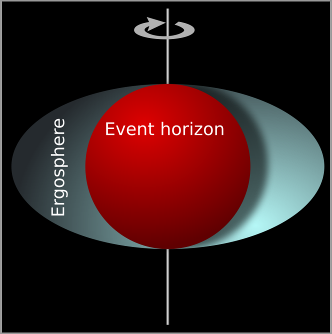
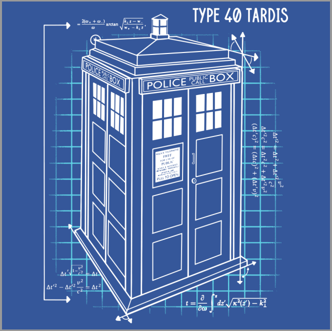


Figure: The Kerr black hole has an ergosphere outside its event horizon.

Kerr Metric

- Between the outer and inner event horizons is a "pocket" of space, larger than it appears from the outside
- In the ergosphere, time and angle switch places
- Rotation becomes as certain as the passage of time



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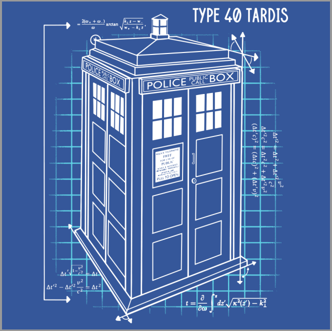
- Penrose process: extract energy from a Kerr black hole by sending a mass into the ergosphere



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Theory of Gravitational Waves

- It is evident from Einstein's equations that wave solutions exist
- However, not at all obvious that these are "true" observable waves
- Feynman helped to establish that gravitational waves do in fact exist and are in principle observable

First Observation: Hulse & Taylor

- Pulsars are fast rotating neutron stars with an extremely precise time period
- When pulsars form binaries, they lose energy to gravitational waves
- Loss of energy leads to a decrease in period, which is observable

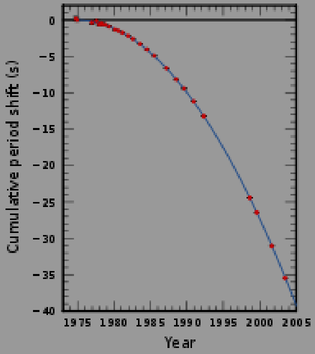


Figure: Hulse & Taylor measured the energy loss from a pulsar binary due to gravitational waves.

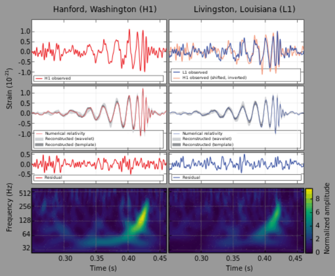
Modern Observation: LIGO

- We can now *directly* measure gravitational waves via their effects on space
- LIGO consists of two identical facilities with 2.5 mile arms in an L shape
- Gravitational waves change the lengths of the arms, causing interference in laser light

Figure: LIGO has measured gravitational waves emitted from merging black holes

GW150914

- On September 15, 2014, following an equipment upgrade, LIGO observed its first gravitational wave event
- The gravitational waves origined 1.4 billion light years away
- Event was a merger of two $30M_\odot$ black holes lasting about 0.2 seconds



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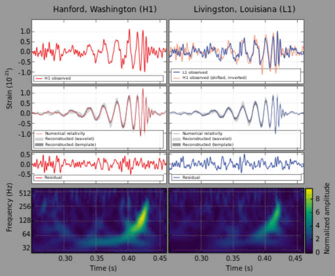
- At the tail end of the event, the power exceeded the power of all stars in the observable universe combined



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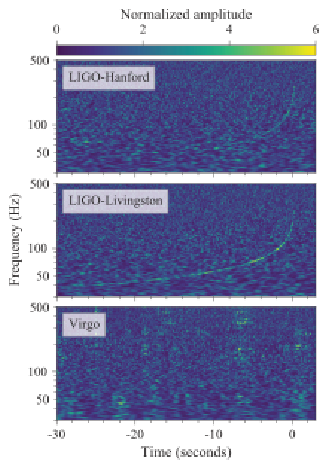
GW170817

- On August 17, 2017, LIGO made the first-ever observation of colliding neutron stars
- Event was accompanied by a short gamma ray burst, confirming a long-standing hypothesis about the origin of these bursts
- Collision occurred nearby, only 130 million light years away



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