

Albert Einstein, Black Holes, and Gravitational Waves

Ross Dempsey

Department of Physics and Astronomy
Johns Hopkins University

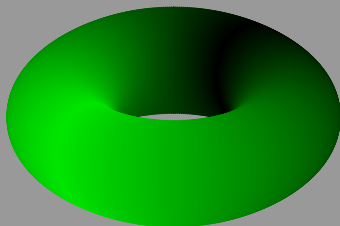
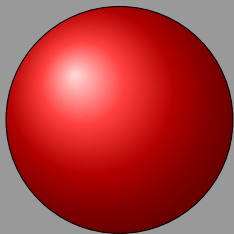
JHU Splash, 2018

Outline

- 1 Curved Space
- 2 Spacetime Metrics
- 3 Einstein's Equations
- 4 Black Holes
- 5 Gravitational Waves

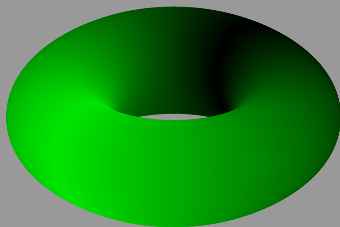
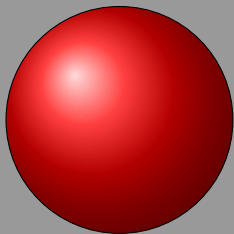
What is curvature?

- Which of the following spaces are curved?



What is curvature?

- Which of the following spaces are curved?



- The sphere is curved, but the torus is not.

Curvature 1: Triangles

- In curved space, the sum of the angles of a triangle does not have to be 180° .
- Try drawing a triangle on the torus: the angles will sum to 180° .
- How about the sphere?

Curvature 1: Triangles

- In curved space, the sum of the angles of a triangle does not have to be 180° .
- Try drawing a triangle on the torus: the angles will sum to 180° .
- How about the sphere?
 - Imagine a triangle between the north pole, the equator at the prime meridian, and the equator at 90° W. What is the sum of angles?

Curvature 1: Triangles

- In curved space, the sum of the angles of a triangle does not have to be 180° .
- Try drawing a triangle on the torus: the angles will sum to 180° .
- How about the sphere?
 - Imagine a triangle between the north pole, the equator at the prime meridian, and the equator at 90° W. What is the sum of angles?
 - Answer: 270°

Curvature 1: Triangles

- In the 1820s, Gauss attempted to measure the curvature of space itself by forming a great triangle between three mountains
- Spoiler: space itself *is* curved. Why did Gauss's measurement not indicate this?

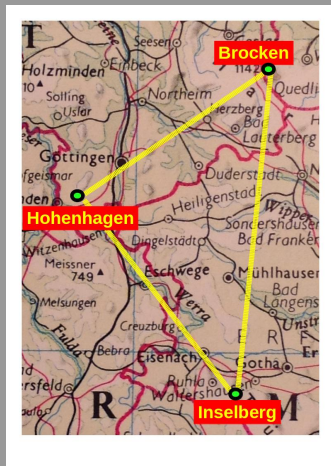


Figure: Gauss measured the angles of a triangle between three mountains.

Curvature 2: Parallel Transport

- Hold your arm out in a “thumbs up.” Can you point your thumb sideways, *without turning your wrist*?

Curvature 2: Parallel Transport

- Hold your arm out in a “thumbs up.” Can you point your thumb sideways, *without turning your wrist*?
- Answer: yes. Use your shoulder.

Curvature 2: Parallel Transport

- Hold your arm out in a “thumbs up.” Can you point your thumb sideways, *without turning your wrist*?
- Answer: yes. Use your shoulder.
- Your thumb indicates a particular direction on the sphere defined by your arm. The motion of your thumb as your shoulder rotates is called *parallel transport*.

Curvature 2: Parallel Transport

- Hold your arm out in a “thumbs up.” Can you point your thumb sideways, *without turning your wrist*?
- Answer: yes. Use your shoulder.
- Your thumb indicates a particular direction on the sphere defined by your arm. The motion of your thumb as your shoulder rotates is called *parallel transport*.
- If parallel transport can change the direction of a vector, space is curved.

Curvature Unified

- The triangle on the sphere had angles 90° too large
- When you moved your thumb around the same triangle, it rotated by 90°

Curvature Unified

- The triangle on the sphere had angles 90° too large
- When you moved your thumb around the same triangle, it rotated by 90°
- Connection: there is a single notion of curvature which leads to both effects

Curvature Unified

- The triangle on the sphere had angles 90° too large
- When you moved your thumb around the same triangle, it rotated by 90°
- Connection: there is a single notion of curvature which leads to both effects
- Curvature is defined at each point. The angle deficit or the parallel transport around a path is related to the curvature contained within that path.

Curvature Unified

- The triangle on the sphere had angles 90° too large
- When you moved your thumb around the same triangle, it rotated by 90°
- Connection: there is a single notion of curvature which leads to both effects
- Curvature is defined at each point. The angle deficit or the parallel transport around a path is related to the curvature contained within that path.
- For a sphere, curvature is constant, so these effects are related to area. What's the area of the spherical triangle?

Gauss-Bonnet Theorem

- The connection between effects on a path and the curvature within that path is spelled out in the Gauss-Bonnet theorem.

Gauss-Bonnet Theorem

- The connection between effects on a path and the curvature within that path is spelled out in the Gauss-Bonnet theorem.
- The formal statement is

$$\int_M K dA + \int_{\partial M} k_g ds = 2\pi \chi(M)$$

\int_M curvature in the area
 $\int_{\partial M}$ effects on boundary
 $\chi(M)$ Euler characteristic

Gauss-Bonnet Theorem

- The connection between effects on a path and the curvature within that path is spelled out in the Gauss-Bonnet theorem.
- The formal statement is

$$\int_M K dA + \int_{\partial M} k_g ds = 2\pi \chi(M)$$

curvature in the area
effects on boundary
Euler characteristic

- Morally: the quantity of interest is the scalar curvature K .

Metrics

- To talk about curvature, we need to start with the geometry of space *at a point*
- Riemann's idea: even curved spaces look flat locally

Metrics

- To talk about curvature, we need to start with the geometry of space *at a point*
- Riemann's idea: even curved spaces look flat locally
- Flat space can be characterized by the distances between points:

$$d(\mathbf{x}; \mathbf{x}^\ell) = \sqrt{(x_1 - x_1^\ell)^2 + (x_2 - x_2^\ell)^2 + (x_3 - x_3^\ell)^2}$$

Metrics

- To talk about curvature, we need to start with the geometry of space *at a point*
- Riemann's idea: even curved spaces look flat locally
- Flat space can be characterized by the distances between points:

$$d(\mathbf{x}; \mathbf{x}^{\prime}) = \sqrt{(x_1 - x_1^{\prime})^2 + (x_2 - x_2^{\prime})^2 + (x_3 - x_3^{\prime})^2}$$

- We care about *local* geometry, not global geometry. So, take nearby points:

$$ds^2 = dx^2 + dy^2 + dz^2$$

Metrics

- To talk about curvature, we need to start with the geometry of space *at a point*
- Riemann's idea: even curved spaces look flat locally
- Flat space can be characterized by the distances between points:

$$d(\mathbf{x}; \mathbf{x}^{\prime}) = \sqrt{(x_1 - x_1^{\prime})^2 + (x_2 - x_2^{\prime})^2 + (x_3 - x_3^{\prime})^2}:$$

- We care about *local* geometry, not global geometry. So, take nearby points:

$$ds^2 = dx^2 + dy^2 + dz^2:$$

- This formula might look different depending on the coordinates. So, generalize:

$$ds^2 = \sum_{i=1}^3 \sum_{j=1}^3 g_{ij} dx^i dx^j:$$

Metrics

- The coefficients g_{ij} constitute the *metric*. In flat Euclidean space with Cartesian coordinates, we know that

$$g_{ij} = \begin{pmatrix} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{pmatrix} :$$

Metrics

- The coefficients g_{ij} constitute the *metric*. In flat Euclidean space with Cartesian coordinates, we know that

$$g_{ij} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \text{ if } i = j$$
$$g_{ij} = 0 \text{ otherwise}$$

- What about the same space with spherical coordinates?

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & r^2 & 0 & 0 \\ 0 & 0 & r^2 \sin^2 \theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Metrics

- The coefficients g_{ij} constitute the *metric*. In flat Euclidean space with Cartesian coordinates, we know that

$$g_{ij} = \begin{pmatrix} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{pmatrix} :$$

- What about the same space with spherical coordinates?

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & r^2 & 0 & 0 \\ 0 & 0 & r^2 \sin^2 \theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- Moral: metrics themselves don't tell us about the space

Curvature in a Metric

- Since the same flat space can have many different metrics, it's not easy to detect when a space itself is curved.
- What if the metric is constant?

$$ds^2 = dx^2 + dx dy + dy^2$$

Curvature in a Metric

- Since the same flat space can have many different metrics, it's not easy to detect when a space itself is curved.
- What if the metric is constant?

$$ds^2 = dx^2 + dx dy + dy^2$$

- A constant metric is always flat space in disguise. In this case, use coordinates $u = \frac{\sqrt{3}}{2}(dx + dy)$, $v = \frac{1}{2}(dx - dy)$. Then

$$\begin{aligned} du^2 + dv^2 &= \frac{3}{4}(dx^2 + 2dx dy + dy^2) + \frac{1}{4}(dx^2 - 2dx dy + dy^2) \\ &= dx^2 + dx dy + dy^2: \end{aligned}$$

Curvature in a Metric

- Since the same flat space can have many different metrics, it's not easy to detect when a space itself is curved.
- What if the metric is constant?

$$ds^2 = dx^2 + dx dy + dy^2$$

- A constant metric is always flat space in disguise. In this case, use coordinates $u = \frac{\sqrt{3}}{2}(dx + dy)$, $v = \frac{1}{2}(dx - dy)$. Then

$$\begin{aligned} du^2 + dv^2 &= \frac{3}{4}(dx^2 + 2dx dy + dy^2) + \frac{1}{4}(dx^2 - 2dx dy + dy^2) \\ &= dx^2 + dx dy + dy^2: \end{aligned}$$

- Moral: curvature must come from changes in the metric

Christoffel Symbols

- The *Christoffel symbols* are related to the rate of change of a metric:

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl} (\partial_l g_{ij} + \partial_i g_{lj} + \partial_j g_{il})$$

- They can be used to compute parallel transport, so they must relate to curvature

Christoffel Symbols

- The *Christoffel symbols* are related to the rate of change of a metric:

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl} (\partial_l g_{ij} + \partial_i g_{lj} + \partial_j g_{il})$$

- They can be used to compute parallel transport, so they must relate to curvature
- Downside: Christoffel symbols depend on coordinate system, like the metric

Riemann Tensor

- Christoffel symbols can be combined to form an object which tells us about curvature, regardless of the underlying coordinate system
- This is the *Riemann curvature tensor*:

$$R^i{}_{jkl} = \partial_k \Gamma^i{}_{lj} - \partial_l \Gamma^i{}_{kj} + \Gamma^i{}_{ka} \Gamma^a{}_{lj} - \Gamma^i{}_{la} \Gamma^a{}_{kj}$$

- If any component of the Riemann tensor is nonzero, the space is curved

Ricci Scalar

- To go from a matrix to a scalar, you can sum its diagonal entries, obtaining the *trace*
- The trace is independent of the basis for the matrix
- Likewise, with the Riemann tensor, we can form a single number which is independent of the coordinate system:

$$R = R^i_i; \quad R_{ij} = R^a_{iaj}.$$

- This number R is called the *Ricci scalar*

Summary

- Metric g_{ij} : tells us how to measure distance near a point
- Christoffel symbol Γ^i_{jk} : tells us how the metric changes
 - Also how to transport vectors, and how to find the shortest path between points
- Ricci tensor R^i_{jkl} : complete, coordinate-independent information about curvature
- Ricci scalar R : a single number indicating curvature

Einstein-Hilbert Action

- Theories of physics can be specified by a single quantity, the *action*
- The action should be sampled uniformly over space and time, so we write it as the integral of a *Lagrangian*
- The Lagrangian is a scalar and must respect the symmetry of the theory
- Einstein tells us (i) spacetime is curved and (ii) coordinate systems don't matter (relativity)

Einstein-Hilbert Action

- Theories of physics can be specified by a single quantity, the *action*
- The action should be sampled uniformly over space and time, so we write it as the integral of a *Lagrangian*
- The Lagrangian is a scalar and must respect the symmetry of the theory
- Einstein tells us (i) spacetime is curved and (ii) coordinate systems don't matter (relativity)
- Natural conclusion: the Lagrangian is the Ricci scalar R of spacetime

Einstein Field Equations

- Using the Lagrangian, it is possible to construct the equation of motion:

$$R - \frac{1}{2}g R = 0:$$

- The right hand side is zero only in a vacuum; in the presence of matter, it is related to the *stress-energy tensor* T :

$$R - \frac{1}{2}g R = 8 G T :$$

- The constant $8 G$ is chosen to ensure agreement with Newton's law of gravity

Newtonian Limit

- How does Einstein's theory line up with Newton's?

Newtonian Limit

- How does Einstein's theory line up with Newton's?
- Newton tells us that $F = ma$ and $F = -mr''$, so $a = -r''$

Newtonian Limit

- How does Einstein's theory line up with Newton's?
- Newton tells us that $F = ma$ and $F = \frac{GMm}{r^2}$, so $a = \frac{GM}{r^2}$
- Newton also tells us that $r^2 = 4GM$

Newtonian Limit

- How does Einstein's theory line up with Newton's?
- Newton tells us that $F = ma$ and $F = \frac{1}{2} m r^{-2}$, so $a = \frac{1}{2} r^{-3}$
- Newton also tells us that $r^2 = 4 G M$
- Einstein tells us:

$$a^i = \frac{1}{2} \frac{1 + g_{tt}}{r^2} \frac{1}{4 G M}$$

Newtonian Limit

- How does Einstein's theory line up with Newton's?
- Newton tells us that $F = ma$ and $F = \frac{GMm}{r^2}$, so $a = \frac{GM}{r^2}$
- Newton also tells us that $r^2 = 4G$
- Einstein tells us:

$$a^i = \frac{1}{2} \frac{1 + g_{tt}}{r^2} \delta^{ij} \delta^{jk} = \frac{1}{2} r^{-2} (1 + g_{tt}) \delta^{ij} \delta^{jk}$$

- Interpret $\frac{1}{2}(1 + g_{tt})$ as $\frac{1}{2}(1 + g_{tt})$, then the theories agree in the weak gravity limit

Meaning of Einstein's Equations

- Consider the source-free Einstein equation, $R - \frac{1}{2}g R = 0$
- R are derivatives of Γ , which are derivatives of g
- Einstein's equations are complicated, nonlinear partial differential equations of the metric
- One solution is flat space, $R_{\mu\nu} = 0$ and $R = 0$
- Differential equations can have many solutions, but finding them is extremely difficult in this case

Schwarzschild Solution

- Shortly after Einstein published his equation, Schwarzschild found a static, spherically symmetric solution:

$$ds^2 = \left(1 - \frac{r_s}{r}\right) dt^2 + \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2;$$

- The radius r_s is called the *Schwarzschild radius*

$$= \frac{1}{2}(1 + g_{tt}) = \frac{r_s}{2} \Rightarrow r_s = 2GM$$

- Note that ds^2 becomes undefined at $r = r_s$. What does this mean?

Black Holes

- The surface $r = r_S$ is an *event horizon*
- What happens to an observer who crosses the event horizon?

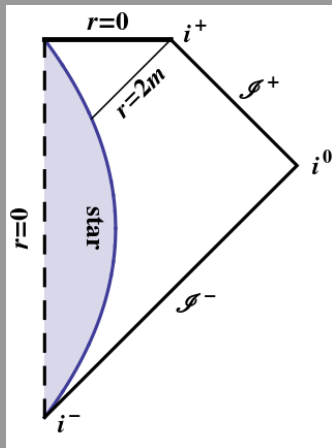


Figure: A Penrose diagram for a star collapsing into a black hole.

Black Holes

- The surface $r = r_s$ is an *event horizon*
- What happens to an observer who crosses the event horizon?
- Observers can only move within a 45 degree cone, so no escape is possible

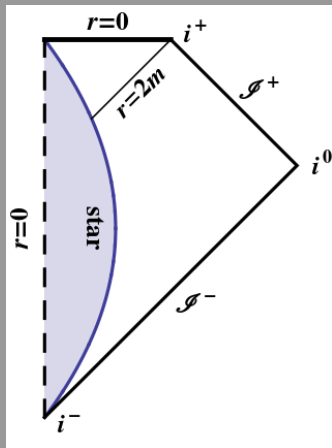


Figure: A Penrose diagram for a star collapsing into a black hole.

Black Holes

- The surface $r = r_S$ is an *event horizon*
- What happens to an observer who crosses the event horizon?
- Observers can only move within a 45 degree cone, so no escape is possible
- The observer will reach $r = 0$ within a finite time, after which *spacetime ends*

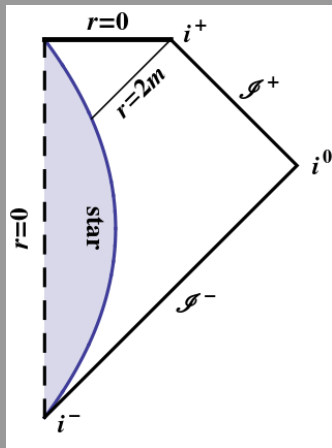


Figure: A Penrose diagram for a star collapsing into a black hole.

Other Black Holes

- Black holes are fully specified by mass, charge, and angular momentum
- Fun fact: existence of charged black holes implies gravity is the weakest force
- Rotating black holes are called Kerr black holes. Kerr metric is very complicated

Kerr Metric

- Black holes are fully specified by mass, charge, and angular momentum
- Rotating black holes are called Kerr black holes. Kerr metric is extremely complicated
- The black hole has inner and outer event horizons
- An *ergosphere* exists outside the event horizons

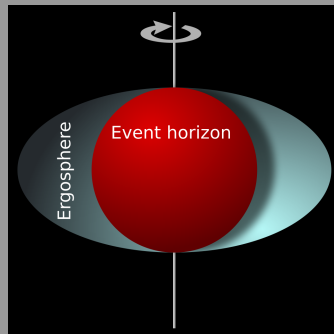


Figure: The Kerr black hole has an ergosphere outside its event horizon.

Kerr Metric

- Between the outer and inner event horizons is a “pocket” of space, larger than it appears from the outside
- In the ergosphere, time and angle switch places
- Rotation becomes as certain as the passage of time

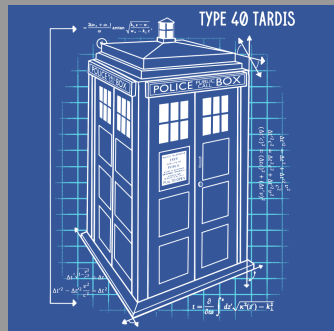


Figure: The Kerr black hole is “bigger on the inside”

Kerr Metric

- Between the outer and inner event horizons is a “pocket” of space, larger than it appears from the outside
- In the ergosphere, time and angle switch places
- Rotation becomes as certain as the passage of time
- Penrose process: extract energy from a Kerr black hole by sending a mass into the ergosphere

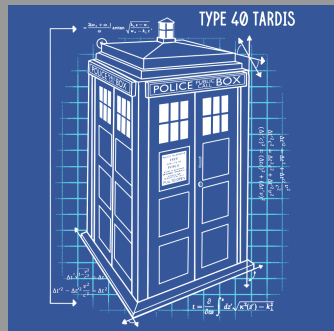


Figure: The Kerr black hole is “bigger on the inside”

Theory of Gravitational Waves

- It is evident from Einstein's equations that wave solutions exist
- However, not at all obvious that these are “true” observable waves
- Feynman helped to establish that gravitational waves do in fact exist and are in principle observable

First Observation: Hulse & Taylor

- Pulsars are fast rotating neutron stars with an extremely precise time period
- When pulsars form binaries, they lose energy to gravitational waves
- Loss of energy leads to a decrease in period, which is observable

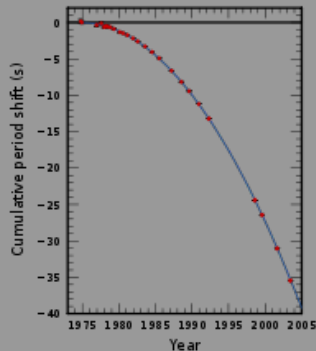


Figure: Hulse & Taylor measured the energy loss from a pulsar binary due to gravitational waves.

Modern Observation: LIGO

- We can now *directly* measure gravitational waves via their effects on space
- LIGO consists of two identical facilities with 2.5 mile arms in an L shape
- Gravitational waves change the lengths of the arms, causing interference in laser light



Figure: LIGO has measured gravitational waves emitted from merging black holes

GW150914

- On September 15, 2014, following an equipment upgrade, LIGO observed its first gravitational wave event
- The gravitational waves originated 1.4 billion light years away
- Event was a merger of two $30M_{\odot}$ black holes lasting about 0.2 seconds

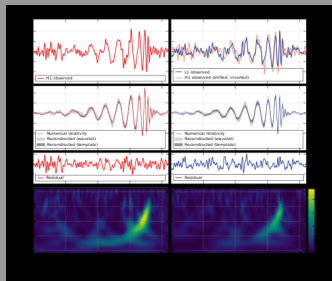


Figure: GW150914, the first observed gravitational wave event

GW150914

- On September 15, 2014, following an equipment upgrade, LIGO observed its first gravitational wave event
- The gravitational waves originated 1.4 billion light years away
- Event was a merger of two $30M_{\odot}$ black holes lasting about 0.2 seconds
- At the tail end of the event, the power exceeded the power of all stars in the observable universe combined

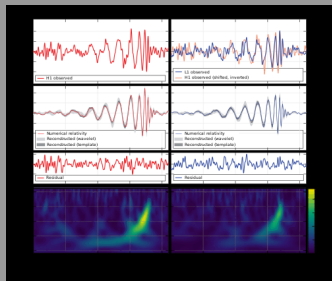


Figure: GW150914, the first observed gravitational wave event

- On August 17, 2017, LIGO made the first-ever observation of colliding neutron stars
- Event was accompanied by a short gamma ray burst, confirming a long-standing hypothesis about the origin of these bursts
- Collision occurred nearby, only 130 million light years away

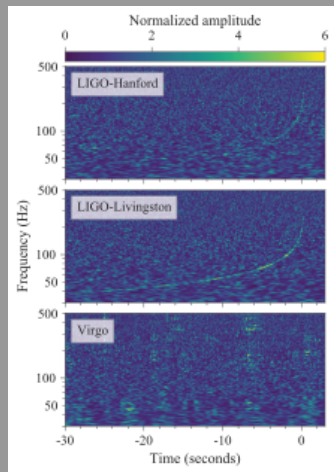


Figure: GW150914, the first observed gravitational wave event