Georg Cantor, Kurt Gödel, and the Incompleteness of Mathematics

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Outline

What is a Set?

- A set is a collection of objects, denoted $S = \left\{1, \nabla, E, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\right\}$
- The objects within a set are known as elements, and are denoted $1 \in S$ or $2 \notin S$
- If every element of T is an element of S, then $T \subseteq S$
  - How many subsets does a set with n elements have?
- The empty set with no elements is denoted $\varnothing$. $\varnothing \subset S$ for every set S
- Union of sets is denoted $A \cup B$; intersection is $A \cap B$

Category of Sets

- A function $f : A \to B$ between two sets A and B specifies an element $f(a) \in B$ for every $a \in A$
- How many functions are there from a set A with n elements to a set B with m elements?
- The *category* of sets consists of all the sets together with all set functions
- An *isomorphism* (or bijection) between two sets A and B consists of:
  - Two functions, $f : A \to B$ and $g : B \to A$
  - Such that $f \circ g = 1_A$ and $g \circ f = 1_B$
- Isomorphism partitions the sets into classes, called *cardinal numbers*

Cardinal Numbers

- Every natural number (0, 1, 2, ...) is a cardinal number
- What about infinite sets? Are they all isomorphic?
  - $\mathbb{N} = \{0, 1, 2, \ldots\}$
  - $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
  - $2\mathbb{Z} = \{\ldots, -4, -2, 0, 2, 4, \ldots\}$
  - $\mathbb{Q} = \{1/2, 68/37, 4/5, \ldots\}$
  - $\mathbb{R} = \{1, e, \sqrt{5}, \ldots\}$

Cardinal of the Continuum

- Georg Cantor proved that the natural numbers $\mathbb{N}$ *are not* isomorphic to the real numbers $\mathbb{R}$
- The argument is called *Cantor diagonalization*. Take any function $f : \mathbb{N} \to \mathbb{R}$. Then:
  - To be a bijection, f must "hit" every real number

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  - But try listing out all the real numbers $f(n)$ in decimal form:

$$f(0) = 1.18736582175$$
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- The cardinal of $\mathbb{R}$ is denoted c, and called the cardinal of the continuum

Infinity of Infinities

- Cantor diagonalization can be used to prove a more general fact
- Let S be a set and 2^S be its *power set*, the set of subsets of S
- Let $f : S \to 2^S$ be any set function. Then construct a subset $T \subset S$ according to:
  - If $s \in f(s)$, $s \notin T$
  - If $s \notin f(s)$, $s \in T$
- Then $T \neq f(s)$ for any $s \in S$
- This means the cardinal of 2^S is greater than the cardinal of S, for any S

Infinity Can't Be That Easy

- Let $S = \{\text{sets not containing themselves}\} = \{x \in \textsc{Set} \mid x \notin x\}$
- Does S contain itself?
  - If $S \in S$, then $S \notin S$
  - If $S \notin S$, then $S \in S$
- Put another way: a barber shaves every man in his town who does not shave themselves. Does the barber shave himself?
- This is *Russell's paradox*. It shows that naïve set theory is inconsistent.

Zermelo-Frankel Set Theory

- Set theory is the foundation of all mathematics. It must be made consistent
- The Zermelo-Frankel (ZF) axioms fix Russell's paradox
- Key idea: build a *universe* of sets in stages: sets, sets containing sets, sets containing sets containing sets, ...

Axiom of Choice

- ZF set theory is often endowed with an additional axiom: the axiom of choice. It becomes ZFC set theory
- The axiom of choice says that, given a collection of sets, it is possible to choose a single element from each set
- Does this seem reasonable?

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  - You can win an infinite hat game
- However, AC is also essential for many fundamental results in mathematics. So we keep it in.

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  - Syntax: rules for forming expressions, called well-formed formulas (wffs)
  - Semantics: rules for understanding a given expression
- Some examples:
  - basic logic (propositional calculus)
  - first-order logic (predicate calculus)
  - Peano arithmetic

Propositional Calculus

- Syntax:
  - Variables $A, B, \ldots$ refer to true (T) or false (F)
  - Symbol $\neg A$ refers to the negation of A
  - Logical connectives: $A \vee B$ (A or B), $A \wedge B$ (A and B), $A \implies B$ (A implies B), $A \iff B$ (A iff B)

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- Semantics:
  - Build up the truth value from the individual pieces. Example:

$$A = \text{it is raining today}$$

$$B = \text{everyone showed up for this class}$$

$$C = \text{Donald Trump is the President of the United States}$$

$$(A \vee C) \wedge (B \implies (\neg A \wedge C)) \wedge (C \iff A)$$

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- Predicate calculus uses expressions like this to better capture "things" and relations between them

Predicate Calculus

- Predicate calculus can use *quantifiers* to make statements about collections of things
  - $\exists \rightarrow$ there exists
  - $\forall \rightarrow$ for all
  - Example: $\forall x(Cx \implies \exists y(L(y,x))$, where
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- Sentences in predicate calculus generally can't be evaluated on their own
  - *Models*: interpretations of variables and predicate symbols
  - Example model: domain = {Donald Trump, United States, Botswana}, C = {United States, Botswana}, L = {(Donald Trump, United States)}
  - Is the above sentence true or false relative to this model?

Practice

- Translate the following sentences into predicate calculus:
  - In every school, there is a bully.
  - If it is raining here, then it is raining in every adjacent city.
  - The best dog is a labrador.
  - Students who get As become lawyers.

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- Predicate calculus is great! Can it express everything?

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- Spoiler: **you can't**
  - Löwenheim-Skolen Theorem: if a theory has an infinite model of cardinality λ, it has models with every cardinality exceeding λ
  - In plain English: if you try to use predicate calculus to pin down the real numbers, you'll inevitably include things you didn't intend
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- Takeaway: logic has its limits. How deep are they?

Interlude: Peano Arithmetic

- The Peano axioms are an attempt to formalize arithmetic on natural numbers, using 0 and S, the successor function $(+1)$
  - (S1) $\forall x(S(x) \neq 0)$
  - (S2) $\forall x \forall y(S(x) = S(y) \implies x = y)$
  - (A1) $\forall x(x + 0 = x)$
  - (A2) $\forall x \forall y(x + S(y) = S(x + y))$
  - (M1) $\forall x(x \cdot 0 = 0)$
  - (M2) $\forall x \forall y(x \cdot S(y) = x \cdot y + x)$
  - (IS) For any formula ϕ, $(\phi(x) \implies \phi(S(x))) \implies (\forall x(\phi(x)))$

Gödel Numbering

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- So: can we translate logical statements about arithmetic into numbers?

Gödel Numbering

- Yes, via Gödel numbering:
  - Associate to each symbol of the system a natural number. For example:

$()$	$0S$	$\neg\vee$	$\wedge \implies$	$\iff \forall$	$\exists =$	...
12	34	56	78	910	11...	

  - Given a formula, such as $\exists x(S(S(x)) = S(S(S(S(0)))))$:
    - Convert each symbol to its number: 10 12 1 4 1 4 1 12 2 2 11 4 1 4 1 4 1 4 1 3 2 2 2 2 2
    - Use these as exponents in a prime factorization:

$$2^{10}3^{12}5^{1}7^{4}11^{1}13^{4}17^{1}19^{12}23^{2}29^{2}31^{11}37^{4}\times$$
$$41^{1}43^{4}47^{1}53^{4}59^{1}61^{4}67^{1}71^{3}73^{2}79^{2}83^{2}89^{2}97^{2}$$

    - Prime factorizations are unique, so every formula gets a unique number

Gödel Numbering

- How about proofs?
  - A proof is a sequence of formulas $\phi_1, \phi_2 \ldots, \phi_n$, such that each statement follows from the previous one.
  - Associate to each statement its Gödel number $\alpha_1, \alpha_2, \ldots, \alpha_n$
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- Now we can build predicates on natural numbers that tell us about the structure of logic
  - Let $G(\phi)$ denote the Gödel number of the sentence ϕ
  - Let $f(x, y)$ be the Gödel number of the formula obtained by replacing occurences of z in the formula labeled by x with the number y
  - Let $\Gamma(x, y)$ indicate that y is the sequence number of a proof ending in the statement with Gödel number x
  - Difficult exercise: translate the following (true) claim into English:

$$\exists x(x = G(\phi) \land \exists y_1(\Gamma(G(\phi), y_1)) \land \exists y_2(\Gamma(G(\neg\phi), y_2)))$$
$$\implies \forall x(\exists y(\Gamma(x, y)))$$

Gödel's First Incompleteness Theorem

- Consider the formula $\neg\exists y(\Gamma(f(z,z),y))$; there does not exist a proof of some statement (the statement numbered by z with occurrences of z replaced by z itself)
- Let the Gödel number of this be i
- Consider the formula $\neg\exists y(\Gamma(f(i,i),y))$ with Gödel number j. Note $j = f(i,i)$
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  - Then $\Gamma(j,k)$ holds, which means $\Gamma(f(i,i),k)$ holds.
  - But then $\exists y(\Gamma(f(i,i),y))$...so the system is inconsistent, a contradiction.

Gödel's First Incompleteness Theorem

- Now consider the opposite sentence, $\exists y(\Gamma f(i,i), y)$
- This is also not provable. Proof:
  - Suppose it is provable. Then $\Gamma(j,k) = \Gamma(f(i,i),k)$ can't hold for any k
  - Thus, for every number k, $\neg\Gamma(f(i,i),k)$ is provable
  - But this contradicts the sentence itself, $\exists y(\Gamma f(i,i), y)$
- Conclusion: there is a sentence such that neither it nor its negation is provable. This sentence is *undecidable*.

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    - One example: Diophantine solvability. Can a given equation be solved in integers?
    - Cool example: mortal matrices. Can a given set of matrices be mulitiplied in some order to give the zero matrix?

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- A natural question: is there a cardinal in between?
- Georg Cantor believed the answer to be no, but couldn't prove this
- Not his fault: in 1963, the continuum hypothesis was proved to be undecidable in ZFC set theory

Cantor's Insanity

- Cantor spent his life in this world of infinity confusions
- The cause, nature, and extent of his mental disorder are not readily apparent, but:
  - He definitely suffered from bipolar disorder
  - He was ostracized from mathematics, largely by Kronecker, which didn't help matters
  - He was the first person to see some of the most mind-boggling facts in mathematics, which definitely didn't help matters

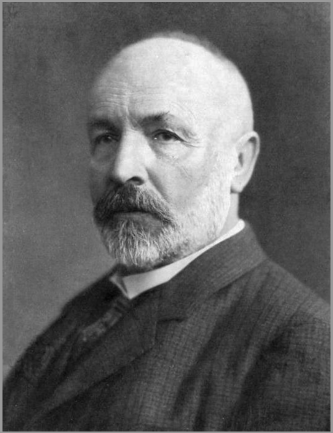

Figure: Georg Cantor, 1845–1918

Gödel's Insanity

- Mathematics was considered the most unimpeachable, unassailable discipline for millennia
- Gödel watched this entire house of cards fall
  - Always a quirky character: citizenship debacle
  - Developed an irrational fear of being poisoned towards the end of his life
  - Eventually starved himself to death

Figure: Kurt Gödel, 1906–1978